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Sun-Yung Alice Chang

Non-linear Elliptic Equations in Conformal Geometry

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Published with the support of the Huber-Kudlich-Stiftung, Zürich

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2000 Mathematics Subject Classification 53Axx; 58Jxx

Bibliographic information published by Die Deutsche Bibliothek

Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;
detailed bibliographic data are available in the Internet at http://dnb.ddb.de.

ISBN 3-03719-006-X

Contact address:

European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C1
CH-8092 Zürich
Switzerland

Phone: +41 (0)1 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Printed on acid-free paper produced from chlorine-free pulp. TCF∞
Printed in Germany
9 8 7 6 5 4 3 2 1

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Preface

Between April and July of 2001, I gave the Nachdiplom lecture series at ETH in Zurich. The lectures concerned the study of some non-linear partial differential equations related to curvature invariants in conformal geometry. A classic example of such a differential equation on a compact surface is the Gaussian curvature equation under conformal change of metrics. On manifolds of dimension four, an analogue of the Gaussian curvature is the Pfaffian integrand in the Gauss-Bonnet formula: on a Riemannian manifold (M, g) of dimension four, denote the Weyl–Schouten tensor A as

$$A_{ij} = R_{ij} - \frac{R}{6} g_{ij}$$

where R_{ij} is the Ricci tensor and R is the scalar curvature of the Riemannian metric g; denote the second elementary symmetric function of A as

$$\sigma_2(A) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}[(TrA)^2 - |A|^2],$$

where λ_i $(1 \leq i \leq 4)$ are the eigenvalues of A; then one has the Gauss Bonnet formula

$$8\pi^2(\chi M) = \int (\frac{1}{4}|W|^2 + \sigma_2(A))dv,$$

where W denotes the Weyl tensor. Under conformal change of metrics, $|W|^2 dv$ is point-wisely conformally invariant, thus$\int \sigma_2(A) dv$ is conformally invariant. The main focus of these lecture notes is the study of the partial differential equation describing the curvature polynomial $\sigma_2(A)$ under conformal change of metrics.

The notes are organized as follows: In Chapters 1 and 2, I discuss the equation prescribing Gaussian curvature on compact surface, provide background, and describe the main analytic tool, Moser–Trudinger inequalities, in the study. In Chapter 3, I describe the connection between Moser–Trudinger inequality to the Polyakov formula for the functional determinant of the Laplacian operator on compact surfaces. In Chapters 4 to 6, I discuss general conformal invariants, the connection of conformal invariants to conformal covariant operators on manifolds of dimension three and higher, with emphasis on a special 4-th operator (called the Paneitz operator) on manifolds of dimension 4. Finally in Chapters 7–10, I study the connection of the Paneitz operator to the curvature polynomial $\sigma_2(A)$ described above. I also report the work of Chang–Gursky–Yang [23] on the existence on manifolds (M^4, g) of solutions with $\sigma_2(A) > 0$ under the assumptions that $\int \sigma_(A) > 0$ and g be of positive Yamabe class.

The lectures were given at an early stage, when the study of the fully non-linear PDEs like that of $\sigma_2(A)$ were first developed. Since then, there has been much progress both in the form of existence and regularity results on such equations. Readers are referred to the article by Gursky–Viaclovsky [56], where a simpler proof, from a somewhat different perspective, of the main result in [23] discussed in these notes is given. There have also been important results on the existence of general conformal invariants by Graham–Zworski [50] and Fefferman–Graham [44]. There is also a more recent survey article [20] for recent developments in this research field.

I wish first to thank Heiko von der Mosel, who originally took the notes that form the basis of this publication. Without his assistance in organizing and correcting, these notes could not have been published. I also wish to thank Meijun Zhu, Fengbo Hang, Paul Yang, Sophie Chen, and Edward Fan for reading the manuscript and making many useful suggestions. Finally, I would like to thank the participants at ETH during the lectures for their input and interest; particular thanks go to Michael Struwe for arranging for a very rewarding visit at ETH.

Alice Chang
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September, 2004

§ 1 Gaussian curvature equation

Let(M^2, g_0) be a compact closed two-dimensional surface with a given metric g_0 and Gaussian curvature K_{g_0}. We are interested in the behavior of the Gaussian curvature under *conformal change of the metric*. That is, we consider the metric

$$\bar{g} := \rho g_0 \tag{1.1}$$

for some $\rho \in C^\infty(M), \rho > 0$. Notice that $\bar{g}$ is conformal to g_0, i.e., while the length of a vector changes; the angle between any two vectors is preserved under the change of metrics from g_0 to $\bar{g}$ on M. From now on we write

$$\bar{g} = g_w := e^{2w} g_0 \tag{1.2}$$

for some function $w \in C^\infty(M)$.

Proposition 1.1 *Let K_{g_w} be the Gaussian curvature of (M^2, g_w). Then*

$$\Delta_0 w + K_{g_w} e^{2w} = K_{g_0}. \tag{1.3}$$

Equation (1.3) is called the *prescribed Gaussian curvature equation*, where $\Delta_0 = \Delta_{g_0}$ denotes the Laplace–Beltrami operator with respect to the background metric g_0. Sometimes we also denote Δ_0 as Δ when the background metric is specified.

Proof of Proposition 1.1. Recall the definition of the Riemann curvature tensor (cf. [3], [86]). For that let $p \in M^n$, and take an orthonormal basis $\{e_i\}$ of the tangent space T_pM of M at p. Then for two vector fields $X, Y \in T_pM$ one has

$$\begin{aligned} R(X,Y) &:= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \\ R(e_i, e_j) &= \nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}, \end{aligned}$$

where the two-form R defines the *curvature* of the Riemannian connection ∇.

The *Christoffel symbols* of g are given by

$$\Gamma^k_{ij} := \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right),$$

and they satisfy

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k.$$

Let $R^l_{kij} := g(R(e_i, e_j)e_k, e_l)$, then the *Ricci tensor* is defined as

$$R_{ij} := R^k_{ikj},$$

and the *scalar curvature* is obtained by contraction again:

$$R := R_{ij} g^{ij}.$$

For $\bar{g} = \rho g_0, \rho > 0$ one computes directly (using $\bar{g}_{il} = \rho(g_0)_{il}, \bar{g}^{kl} = \rho^{-1} g_0^{kl}$), that the Christoffel symbols Γ_{ij}^k of $\bar{g}$ satisfy

$$\bar{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2}\left(\delta_i^k \frac{\partial \log \rho}{\partial x^j} + \delta_j^k \frac{\partial \log \rho}{\partial x^i} - g^{kl} g_{ij} \frac{\partial \log \rho}{\partial x^l}\right).$$

When $n = 2$ we write $\rho = e^{2w}$ and get after a lengthy calculation

$$\bar{R}_{1212} = e^{-2w}((R_{g_0})_{1212} - 2\Delta_0 w),$$

which is equivalent to (1.3), since $K_{g_0} = \frac{1}{2}(R_{g_0})_{1212}$ and $K_{g_w} = \frac{1}{2}\bar{R}_{1212}$. $\square$

Remark 1.2 Integrating both sides of (1.3) over M gives in case M is orientable

$$\begin{aligned}
\int_M K_{g_0}\, dv_0 &= \int_M K_{g_w} e^{2w}\, dv_0 \\
&= \int_M K_{g_w}\, dv_{g_w} \\
&= 2\pi\chi(M) \\
&= 2\pi(2 - 2ge),
\end{aligned} \tag{1.4}$$

where $dv_0 = dv_{g_0}$, $\chi(M)$ is the Euler characteristic and ge the genus of M. Here we used the Gauss–Bonnet Theorem. Hence $\int K_g\, dv_g$ is conformally invariant, and its sign is determined by the sign of $\chi(M)$.

One of the central problems is: Given a function $K \in C^\infty(M)$ on a compact closed two-dimensional manifold M with fixed background metric g_0, when does there exist a metric $\bar{g}$ conformal to g_0, such that

$$K_{\bar{g}} = K?$$

In other words, does (1.3) admit a solution w, such that $K_{g_w} = K$? This is usually called the problem of "prescribing Gaussian curvature". In the case when the compact surface is the standard 2-sphere, the problem is commonly attributed to L. Nirenberg and is called the "Nirenberg" problem.

Kazdan and Warner [59] gave some necessary and sufficient conditions for the existence of solutions for (1.3) in some cases.

Theorem 1.3 *Let* $\chi(M) = 0$. *Then* (1.3) *has a solution* w *iff either* (i) $K \equiv 0$ *or* (ii) K *changes sign with* $\int_M K e^{2f}\, dv_0 < 0$, *where* f *is a solution of* $\Delta_0 f = K_{g_0}$.

Proof. By (1.4) and the assumption $\chi(M) = 0$, we have

$$0 = \int_M K_{g_0}\, dv_0 = \int_M K_{g_w}\, dv_{g_w}, \tag{1.5}$$

hence $\Delta_0 f = K_{g_0}$ is solvable on M. Moreover, f is unique up to a constant. If w solves (1.3), then one easily checks that $u\colon = w - f$ is a solution of

$$\Delta_0 u + Ke^{2(u+f)} = 0, \tag{1.6}$$

which implies by integration

$$\begin{aligned} \int_M Ke^{2f}\, dv_0 &= -\int_M (\Delta_0 u) e^{-2u}\, dv_0 \\ &= \int_M \nabla_0 u \cdot \nabla_0 (e^{-2u})\, dv_0 \\ &= -2 \int_M |\nabla_0 u|^2 e^{-2u}\, dv_0 \le 0. \end{aligned} \tag{1.7}$$

Equality occurs iff $|\nabla_0 u| \equiv 0$, which implies that $u \equiv$ const., i.e., $\Delta_0 u \equiv 0$, hence by (1.6) $K \equiv 0$. If $K \not\equiv 0$, on the other hand, we have $\int Ke^{2f}\, dv_0 < 0$, and we infer from (1.5) that K changes sign. This proves necessity.

If $K \equiv 0$, then $w := f$ with $\Delta_0 f = K_{g_0}$ solves (1.3). If $K \not\equiv 0, K$ changes sign and $\int_M Ke^{2f}\, dv_0 < 0$, then we claim that we can find a solution u of equation (1.6), which also solves (1.3) setting $w := u + f$ as seen above.

To prove this claim consider the set

$$\mathcal{C}\colon = \{u \in W^{1,2}(M)\colon \int_M Ke^{2(u+f)}\, dv_0 = 0 \text{ and } \int_M u dv_0 = 0\},$$

which is not empty, since K changes sign by assumption.

If we find a minimizing function $u_0 \in \mathcal{C}$ of the energy functional

$$E(u)\colon = \frac{1}{2} \int_M |\nabla_0 u|^2\, dv_0,$$

i.e., with

$$E(u_0) = \inf_{u \in \mathcal{C}} E(u), \tag{1.8}$$

then there exist some Lagrange multipliers $\alpha, \beta \in \mathbb{R}$, such that

$$\Delta_0 u_0 + \alpha + \beta K e^{2(u_0+f)} = 0 \quad \text{on } M. \tag{1.9}$$

Integrating this equation over M we immediately obtain $\alpha = 0$ by the first integral constraint in the definition of $\mathcal{C}$.

By the same argument we obtain for β,

$$\begin{aligned} \beta \int_M Ke^{2f}\, dv_0 &= -\int e^{-2u_0} \Delta_0 u_0\, dv_0 \\ &= \int \nabla_0 (e^{-2u_0}) \cdot \nabla_0 u_0\, dv_0 \\ &= -2 \int |\nabla_0 u_0|^2 e^{-2u_0}\, dv_0 < 0, \end{aligned}$$

which by our assumption $\int_M Ke^{2f}\,dv_0 < 0$ means that $\beta > 0$. Thus the shift $v_0 \colon = u_0 + \frac{1}{2}\log\beta$ satisfies

$$\Delta_0 v_0 + Ke^{2(v_0+f)} = 0 \quad \text{on } M \tag{1.10}$$

as a consequence of (1.9) with $\alpha = 0$.

To justify the above arguments involving the Euler–Lagrange equation pointwise on M, we need to show that any minimizer of $E(\cdot)$ in $\mathcal{C}$ is sufficiently smooth to carry out the differentiation. In fact, it will be shown below (see Corollary 1.7), that for all $v \in W^{1,2}(M)$ with finite energy $E(v) < \infty$ one obtains

$$e^v \in L^p(M) \text{ for all } p > 1. \tag{1.11}$$

This implies that $\Delta_0 v_0 \in L^p(M)$ for all $p > 1$ by (1.10), in particular $v_0 \in C^\infty(M)$ by standard elliptic estimates.

It remains to show that a minimizer $u_0 \in \mathcal{C}$ satisfying (1.8) actually exists. Taking a minimal sequence $\{u_i\}_{i\in\mathbb{N}} \subset \mathcal{C}, E(u_i) \to \inf_{u\in\mathcal{C}} E(u)$ as $i \to \infty$, we readily get weak convergence $u_i \rightharpoonup u_0 \in W^{1,2}(M)$ with

$$E(u_0) \le \liminf_{i\to\infty} E(u_i) = \inf_{u\in\mathcal{C}} E(u). \tag{1.12}$$

Hence

$$0 = \int_M u_i\,dv_0 \to \int_M u_0\,dv_0 \quad \text{for } i \to \infty,$$

and we will see later (Corollary 1.8) that also

$$0 = \int_M Ke^{2(u_i+f)}\,dv_0 \to \int_M Ke^{2(u_0+f)}\,dv_0 \quad \text{as } i \to \infty, \tag{1.13}$$

which shows $u_0 \in \mathcal{C}$. Thus by (1.12)

$$\inf_{u\in\mathcal{C}} E(u) \le E(u_0) \le \inf_{u\in\mathcal{C}} E(u) \Rightarrow E(u_0) = \inf_{u\in\mathcal{C}} E(u),$$

which concludes the proof of Theorem 1.3. $\square$

Now we are going to provide the analytical tools necessary to prove (1.11) and (1.13).

Recall Sobolev's embedding theorem, which states that for a domain $\Omega \subset \mathbb{R}^n$ one has $W_0^{\alpha,q}(\Omega) \hookrightarrow L^p(\Omega)$ for $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$, $q\alpha < n$.

If $\alpha = 1, n = 2, q < 2$ we obtain $W_0^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$. In general one cannot take the limits $q \to 2, p \to \infty$, i.e.,

$$W_0^{1,2}(\Omega) \not\hookrightarrow L^\infty(\Omega),$$

as one can see for the function $u(x) \colon = \log(1 + \log\frac{1}{|x|})$ on $B_1(0) \subset \mathbb{R}^2$.

Instead N. Trudinger proved exponential L^2-integrability in the following sense.

Proposition 1.4 [87] *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain and $u \in W_0^{1,2}(\Omega)$ with $\int_\Omega |\nabla u|^2\,dx \le 1$. Then there exist universal constants $\beta > 0, C_1 > 0$, such that*

$$\int_\Omega e^{\beta u^2}\,dx \le C_1|\Omega|, \tag{1.14}$$

and we write $W_0^{1,2}(\Omega) \hookrightarrow e^{L^2}(\Omega)$.

Remark 1.5 Under the assumption $\int_\Omega |\nabla u|^2\,dx \le 1$ the inequality (1.14) is equivalent to the following:

There is a universal constant $C_2 > 0$, such that

$$||u||_{L^p(\Omega)} \le C_2\sqrt{p}|\Omega|^{\frac{1}{p}} \text{ for all } p \ge 2. \tag{1.15}$$

Let us prove this remark first.

"$\Rightarrow$" For all $k \in \mathbb{N}$ one has

$$\frac{1}{k!}\int_\Omega (\beta u^2)^k\,dx \le C_1|\Omega|,$$

hence

$$\begin{aligned}\left(\int_\Omega u^{2k}\,dx\right)^{\frac{1}{2k}} &\le \left(\frac{k!}{\beta^k}C_1|\Omega|\right)^{\frac{1}{2k}} \\ &= (k!)^{\frac{1}{2k}}\frac{1}{\sqrt{\beta}}C_1^{\frac{1}{2k}}|\Omega|^{\frac{1}{2k}} \\ &\le \tilde{C}_2\sqrt{2k}|\Omega|^{\frac{1}{2k}},\end{aligned}$$

since $(k!)^{\frac{1}{k}} \le k$. This proves the claim for $p:=2k, k \in \mathbb{N}$. For odd p a simple use of Hölder's inequality gives

$$\begin{aligned}\left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}} \le \left(\int_\Omega u^{2p}\,dx\right)^{\frac{1}{2p}}|\Omega|^{\frac{1}{2p}} &\le \tilde{C}_2\sqrt{2p}|\Omega|^{\frac{1}{2p}}\cdot|\Omega|^{\frac{1}{2p}} \\ &=: C_2\sqrt{p}|\Omega|^{\frac{1}{p}}.\end{aligned}$$

"$\Leftarrow$"

$$\begin{aligned}\int_\Omega e^{\beta u^2}\,dx &= \int_\Omega \sum_{k=0}^\infty \frac{1}{k!}(\beta|u|^2)^k\,dx \\ &= \sum_{k=0}^\infty \frac{\beta^k}{k!}||u||^{2k}_{L^{2k}(\Omega)} \\ &\le \sum_{k=0}^\infty \frac{\beta^k}{k!}\left[C_2\sqrt{2k}|\Omega|^{\frac{1}{2k}}\right]^{2k} \\ &= \sum_{k=0}^\infty \frac{1}{k!}(2\beta C_2^2 k)^k|\Omega| \le C_1|\Omega|,\end{aligned}$$

if one chooses β so small that $2\beta C_2^2 < e^{-1}$, which according to Stirling's formula implies that the infinite series $\sum_{k=0}^{\infty} \frac{1}{k!}(2\beta C_2^2 k)^k$ is finite.

Proof of Proposition 1.4. Using the previous remark, it suffices to show (1.15). By symmetric rearrangement[1] and scaling we may take $\Omega\colon = B_1(0) \subset \mathbb{R}^2$. Furthermore, we may assume $u \in \mathbb{C}^\infty$.

We can represent u as

$$u(x) = -\frac{1}{2\pi}\int_{B_1(0)} \Delta u(y) \log|x-y|\, dy,$$

which after integration by parts leads to the estimate

$$\begin{aligned}|u(x)| &\leq C\int_{B_1(0)} |\nabla u(y)||x-y|^{-1}\, dy\\ &\leq C\left(\int_{B_1(0)} |\nabla u(y)|^2|x-y|^{-a}\, dy\right)^{\frac{1}{p}}\left(\int_{B_1(0)} |x-y|^{-a}\right)^{\frac{1}{2}}\left(\int_{B_1(0)} |\nabla u(y)|^2\, dy\right)^{\frac{1}{2}-\frac{1}{p}},\end{aligned}$$

using Hölder's inequality for $\frac{a}{p} + \frac{a}{2} = 1$.

Now $\int_{B_1(0)} |x-y|^{-a}\, dy$ is finite, since for $x, y \in B_1(0)$ one has $B_1(0) \subset B_2(x)$ and then

$$\int_{B_1(0)} |x-y|^{-a}\, dy \leq \int_{B_2(x)} |x-y|^{-a}\, dy = C\left[\frac{r^{2-a}}{2-a}\right]_{r=0}^{r=2} \leq C(p+2). \tag{1.16}$$

Consequently,

$$\begin{aligned}\int_{B_1(0)} |u|^p\, dx &\leq C\left[\int_{B_1(0)}\int_{B_1(0)} |\nabla u(y)|^2|x-y|^{-a}\, dy\, dx\right]||\nabla u||^{p-2}_{L^2(B_1(0))}(p+2)^{\frac{p}{2}}\\ &\leq ||\nabla u||^p_{L^2(B_1(0))}(p+2)^{\frac{p}{2}+1},\end{aligned}$$

where we used Fubini's Theorem and (1.16) to obtain the last inequality. By assumption $||\nabla u||_{L^2(B_1(0))} \leq 1$, i.e., we have

$$||u||_{L^p(B_1(0))} \leq C_2\sqrt{p}$$

for some universal constant $C_2 > 0$. □

Corollary 1.6 *Let (M^2, g) be compact and closed. Then there exist constants $\beta = \beta(g) > 0$ and $C = C(g) > 0$, such that for all $u \in W^{1,2}(M)$ with*

$$\int_M u\, dv_g = 0, \int_M |\nabla_0 u|^2\, dv_g \leq 1$$

[1] $\int_\Omega e^{\beta u^2}\, dx \leq \int_{B_1(0)} e^{\beta(u^*)^2}\, dx$ and $\int_{B_1(0)} |\nabla u^*|^2\, dx \leq \int_\Omega |\nabla u|^2\, dx$, if u^* is the symmetric rearrangement of u, see [78].

one has

$$\int_M e^{\beta u^2}\,dv_g \le C\,\mathrm{vol}(M,g). \tag{1.17}$$

Proof. Take a partition of unity (U_i,ϕ_i) of M, such that each U_i is diffeomorphic to the unit ball $B_1(0)\subset\mathbb{R}^2$ with $0\le\phi_i\le 1, \phi_i\in C_0^\infty(U_i), \sum_i\phi_i\equiv 1$ on M, and set $u_i\colon=\phi_i u$. Then $\nabla u_i=(\nabla u)\phi_i+(\nabla\phi_i)u$, and by Proposition 1.4 we have

$$||u_i||_{L^p(U_i)}\le \tilde{C}_2\sqrt{p}||\nabla u_i||_{L^2(U_i)}(\mathrm{vol}(U_i))^{\frac{1}{p}} \text{ for } p>2.$$

Hence

$$\begin{aligned}||u||_{L^p(M)} &\le \sum_i ||u_i||_{L^p(U_i)} \le \tilde{C}_2\sqrt{p}(\mathrm{vol}(M,g))^{\frac{1}{p}}\sum_i||\nabla u_i||_{L^2(U_i)}\\ &\le \tilde{\tilde{C}}_2\sqrt{p}(\mathrm{vol}(M,g))^{\frac{1}{p}}(||\nabla u||_{L^2(M)}+||u||_{L^2(M)})\\ &\le C(g)\sqrt{p}(\mathrm{vol}(M,g))^{\frac{1}{p}}||\nabla u||_{L^2(M)},\end{aligned}$$

where we used Poincaré's Inequality, which is valid, since $\int_M u\,dv_g=0$. Notice that $C=C(g)$ depends on the metric g via the partition of unity, in particular the terms involving $\nabla\phi_i$. $\square$

Corollary 1.7 *For a compact and closed manifold (M^2,g) there are constants $\eta>0$ and $c=c(g)$, such that for each $p\ge 2$,*

$$\int_M e^{p(w-\overline{w})}\,dv_g \le c\exp\left[\eta\frac{p^2}{4}||\nabla w||^2_{L^2(M)}\right] \tag{1.18}$$

for all $w\in W^{1,2}(M)$, where

$$\overline{w}\colon=\fint_M w\,dv_g=\frac{1}{\mathrm{vol}(M,g)}\int_M w\,dv_g.$$

Proof. By Young's inequality we get, for $||\nabla w||_{L^2(M)}\neq 0$,

$$p(w-\overline{w})\le\beta\frac{(w-\overline{w})^2}{||\nabla w||^2_{L^2(M)}}+\frac{1}{\beta}\frac{p^2}{4}||\nabla w||^2_{L^2(M)},$$

where $\beta>0$ is the constant of Corollary 1.6. Taking the exponential of this inequality and integrating one obtains for $u\colon=\frac{w-\overline{w}}{||\nabla w||_{L^2(M)}}$ $(\Rightarrow \overline{u}=0$ and $||\nabla u||_{L^2(M)}\le 1)$

$$\begin{aligned}\int_M e^{p(w-\overline{w})}\,dv_g &\le \int_M e^{\beta u^2}\cdot e^{\frac{1}{\beta}\frac{p^2}{4}||\nabla w||^2_{L^2(M)}}\,dv_g\\ &\le \exp\left[\frac{1}{\beta}\frac{p^2}{4}||\nabla w||^2_{L^2(M)}\right]\cdot c(g)\,\mathrm{vol}(M,g),\end{aligned}$$

which concludes the proof if one sets $\eta\colon=\beta^{-1}$ and $c\colon=c(g)\,\mathrm{vol}(M,g)$. $\square$

Corollary 1.8 *If $u_i \rightharpoonup u$ in $W^{1,2}(M)$ as $i \to \infty$, and*

$$\int_M |\nabla u|^2 \, dv_g \le c, \int_M |\nabla u_i|^2 \, dv_g \le c \quad \text{with} \quad \int_M u_i \, dv_g = 0 \quad \text{for all} \quad i \in \mathbb{N},$$

then for each $f \in L^\infty(M)$,

$$\int_M f e^{pu_i} \, dv_g \to \int_M f e^{pu} \, dv_g \text{ as } i \to \infty. \tag{1.19}$$

Proof. Using the simple estimate $|e^x - 1| \le |x|e^{|x|}$ we can write

$$\begin{aligned}
\int_M |e^{pu_i} - e^{pu}| \, dv_g &= \int_M e^{pu}(e^{p(u_i-u)} - 1) \, dv_g \\
&\le \int_M e^{pu} p|u_i - u| e^{p|u_i-u|} \, dv_g \\
&\le C \left(\int_M e^{4pu} dv_g \right)^{\frac{1}{4}} \left(\int_M |u_i - u|^2 dv_g \right)^{\frac{1}{2}} \left(\int_M e^{4p|u_i-u|} \, dv_g \right)^{\frac{1}{4}},
\end{aligned}$$

using Hölder's inequality. The right-hand side tends to zero as $i \to \infty$, since the middle term does by Rellich's theorem, and the two integrals involving exponential terms stay bounded according to (1.18). $\square$

Remark. The case $\chi(M) < 0$ has also been considered by Kazdan and Warner ([59]), but is not completely settled. There are necessary conditions and also sufficient conditions, but a complete characterization of the solvability of the Gaussian curvature equation (1.3) as in Theorem 1.3 remains an open problem for $\chi(M) < 0$. Let us now turn to the case $\chi(M) > 0$.

§ 2 Moser–Trudinger inequality (on the sphere)

When $\chi(M) > 0$, then either $\chi(M) = 2$, in which case M is diffeomorphic to the sphere S^2, or $\chi(M) = 1$, i.e., $M \cong \mathbb{R}P^2$, the real projective space.

Consider $(M, g) := (S^2, g_c)$ with the canonical metric g_c and Gaussian curvature $K_{g_c} \equiv 1$. The Gaussian curvature equation (1.3) then reads as

$$\Delta w + K e^{2w} = 1 \quad \text{on } (S^2, g_c), \tag{2.1}$$

where we denote $\Delta = \Delta g_c$ as before. Here, $K \in C^\infty(S^2)$ is a given function.

Theorem 2.1 [59] *Let $w \in W^{1,2}(S^2)$ be a solution of* (2.1)*. Then*

$$\int_{S^2} \langle \nabla K, \nabla \varphi \rangle e^{2w} \, dv_{g_c} = 0, \tag{2.2}$$

where φ is any of the first eigenfunctions of Δ on the sphere, i.e.,

$$\Delta \varphi + 2\varphi = 0 \quad \text{on } S^2. \tag{2.3}$$

($\varphi = \tilde{\varphi}_{|S^2}$ for $\tilde{\varphi} : \mathbb{R}^3 \to \mathbb{R}$, $\tilde{\varphi}(x) = \sum_{i=1}^3 c_i x^i$, for some real constants c_i, $i = 1, 2, 3$.)

Remark 2.2 By the Gauss–Bonnet Theorem

$$\int_{S^2} K e^{2w} \, dv_{g_c} = 4\pi, \text{ hence } K > 0$$

somewhere on S^2. But this information is not sufficient for the existence of solutions for (2.1). In fact, for $K = K_\varepsilon := 1 + \varepsilon\varphi, \varepsilon \downarrow 0$, the Kazdan–Warner condition (2.2) is violated for every $\varepsilon > 0$, which means that there are functions K arbitrarily close to 1, for which (2.1) is not solvable.

Proof of Theorem 2.1. One has $\nabla_k \nabla_l \tilde{\varphi} = \tilde{\varphi} g_{kl}$ for $\tilde{\varphi}(x) = x_i$ on S^2, hence (2.3) implies

$$2\nabla_k \nabla_l \varphi = \Delta \varphi g_{kl} \quad \text{for } \varphi = \tilde{\varphi}_{|S^2}. \tag{2.4}$$

Integrating by parts repeatedly, and inserting (2.1) and (2.3) we compute

$$\begin{aligned}
\int_{S^2} \langle \nabla K, \nabla \varphi \rangle e^{2w} \, dv_{g_c} &= -\int_{S^2} K \Delta \varphi e^{2w} \, dv_{g_c} - 2\int_{S^2} K \langle \nabla \varphi, \nabla w \rangle e^{2w} \, dv_{g_c} \\
&\underset{(2.1)}{=} -\int_{S^2} \Delta \varphi (1 - \Delta w) \, dv_{g_c} - 2\int_{S^2} \langle \nabla \varphi, \nabla w \rangle (1 - \Delta w) \, dv_{g_c} \\
&\underset{(2.3)}{=} 2\int_{S^2} \varphi (1 - \Delta w) \, dv_{g_c} + 2\int_{S^2} \varphi \Delta w \, dv_{g_c} + 2\int_{S^2} \langle \nabla \varphi, \nabla w \rangle \Delta w \, dv_{g_c} \\
&\underset{(2.3)}{=} -\int_{S^2} \Delta \varphi \, dv_{g_c} + 2\int_{S^2} \nabla_i \varphi \nabla_i w \Delta w \, dv_{g_c}
\end{aligned}$$

$$
\begin{aligned}
&= -2\int_{S^2} \nabla_l(\nabla_i\varphi\nabla_i w)\nabla_l w\, dv_{g_c}\\
&= -2\int_{S^2} \nabla_l\nabla_i\varphi\nabla_i w\nabla_l w\, dv_{g_c} - 2\int_{S^2} \nabla_i\varphi\nabla_l\nabla_i w\nabla_l w\, dv_{g_c}\\
&\underset{(2.4)}{=} -\int_{S^2} g_{li}\nabla_i w\nabla_l w(\Delta\varphi)\, dv_{g_c} - \int_{S^2} \nabla_i\varphi\nabla_i(\nabla_l w\nabla_l w)\, dv_{g_c}\\
&= -\int_{S^2} |\nabla w|^2\Delta\varphi\, dv_{g_c} + \int_{S^2} \Delta\varphi|\nabla w|^2\, dv_{g_c}\\
&= 0. \qquad \square
\end{aligned}
$$

A sufficient condition for the solvability of (2.1) was given by Moser in [64], see also [65].

Theorem 2.3 [Moser] *If $K(-\xi) = K(\xi)$ for all $\xi \in S^2$, and if $\max_{S^2} K > 0$, then* (2.1) *has a solution $w \in C^\infty(S^2)$ with*

$$w(-\xi) = w(\xi) \text{ for all } \xi \in S^2.$$

Sketch of the proof. We consider a variational approach using the functional

$$J_K[w] := \log \fint_{S_2} Ke^{2w}\, dv_{g_c} - \frac{1}{4\pi}\int_{S^2} |\nabla w|^2 dv_{g_c} - 2\fint_{S^2} w\, dv_{g_c}, \tag{2.5}$$

whose critical points, i.e., $w \in W^{1,2}(S^2)$ satisfy the equation[2]

$$-\Delta w + 1 = \frac{Ke^{2w}}{\fint_{S^2} Ke^{2w}\, dv_{g_c}} \quad \text{on } S^2. \tag{2.6}$$

Then the shifted function

$$\tilde{w} := w - \frac{1}{2}\log \fint_{S^2} Ke^{2w}\, dv_{g_c}$$

solves (2.1).

Consequently, the proof boils down to showing the existence of a critical point for the functional $J_K[\cdot]$. For that we need some sharpened versions of Proposition 1.4, Corollary 1.6 and Corollary 1.7. We are going to state these results without proof.

Theorem 2.4 [Moser–Trudinger inequality] *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n\, dx \le 1$. Then there is a constant $C = C(n)$, such that*

$$\int_\Omega e^{\alpha|u|^p}\, dx \le C|\Omega|, \tag{2.7}$$

where $p = \frac{n}{n-1}, \alpha \le \alpha_n := nw_{n-1}^{\frac{1}{n-1}}, w_k :=$ k-dimensional surface measure of S^k.

[2] We have seen before that $W^{1,2}$-solutions of (2.6) are in fact of class $C^\infty(S^2)$; compare with the proof of Theorem 1.3.

Remark 2.5 For $n = 2$ one has $p = 2, \alpha_2 = 2w_1 = 4\pi$. Moser has shown that the constant α_n in the theorem is sharp in contrast to the constant β in Proposition 1.4. In fact, he constructed a sequence $u_k \in W_0^{1,n}(B_1(0))$ with $\int_{B_1(0)} |\nabla u_k|^n \, dx \leq 1$ such that

$$\int_{B_1(0)} e^{\alpha |u_k|^p} \, dx \to \infty \quad \text{as } k \to \infty,$$

if $\alpha > \alpha_n$.

We have seen in Corollary 1.6 that for general compact closed (M, g) the constant on the right-hand side of (1.17) depends on the metric g. Working on (S^2, g_c) allows us to control the constants.

Theorem 2.6 [Moser] *There is a universal constant $C_1 > 0$, such that for all $w \in W^{1,2}(S^2)$ with $\int_{S^2} |\nabla w|^2 \, dv_{g_c} \leq 1$ and $\int_{S^2} w \, dv_{g_c} = 0$,*

$$\int_{S^2} e^{4\pi w^2} \, dv_{g_c} \leq C_1. \tag{2.8}$$

In the same way as we deduced Corollary 1.7 from Corollary 1.6 one can show

Corollary 2.7 *For $C_2 := \log C_1 + \log \frac{1}{4\pi}$,*

$$\log \fint_{S^2} e^{2w} \, dv_{g_c} \leq \left[\frac{1}{4\pi} \int_{S^2} |\nabla w|^2 \, dv_{g_c} + 2 \fint_{S^2} w dv_{g_c} \right] + C_2 \tag{2.9}$$

for all $w \in W^{1,2}(S^2)$.

Remark 2.8 For w as in Theorem 2.6 with $w \not\equiv 0$ one easily gets

$$4\pi = \int_{S^2} dv_{g_c} < \int_{S^2} e^{4\pi w^2} \, dv_{g_c} \leq C_1,$$

hence $C_2 > 0$. For a domain in the plane(i.e., $n = 2$ in Theorem 2.4), Carleson and Chang [19] have proved the existence of an extremal function for the Moser–Trudinger inequality for Theorem 2.4, and the best constant C_2 in the statement of Theorem 2.4 is $> 1 + e$. This result was extended by T.L. Soong [84] proving the existence of extremal functions for (2.8) in Theorem 2.6, see also the results on the structural behavior of such extremal functions in M. Flucher's work, [45]. These investigations are also related to work of A. Beurling on the boundary behavior of analytic functions on the disk, [8]. With different arguments we will need to prove later that $C_2 = 0$ is the best constant in (2.9), which is the content of Onofri's inequality, Theorem 2.11. For even functions on S^2, Moser improved his result, Theorem 2.6:

Theorem 2.9 [Moser] *If* $w \in W^{1,2}(S^2)$ *with* $\displaystyle\fint_{S^2} w\, dv_{g_c} = 0, \int_{S^2} |\nabla w|^2\, dv_{g_c} \le 1$ *and* $w(\xi) = w(-\xi)$ *for almost all* $\xi \in S^2$*, then*

$$\int_{S^2} e^{8\pi w^2}\, dv_{g_c} \le C_3. \tag{2.10}$$

Again we infer

Corollary 2.10 *For* $C_4 := \log C_3 + \log \frac{1}{4\pi}, a = \frac{1}{2}$,

$$\log \fint_{S^2} e^{2w}\, dv_{g_c} \le \left[a \cdot \frac{1}{4\pi} \int_{S^2} |\nabla w|^2\, dv_{g_c} + 2 \fint_{S^2} w\, dv_{g_c} \right] + C_4. \tag{2.11}$$

Let us point out that only $a < 1$ *is crucial for later applications.*

Now we finally turn to the proof of Theorem 2.3:

Proof of Theorem 2.3. Since $K > 0$ somewhere, and K is even,

$$\mathcal{C} := \{ w \in W^{1,2}(S^2) : \int_{S^2} K e^{2w}\, dv_{g_c} > 0, w \text{ even a.e. } \} \ne \emptyset. \tag{2.12}$$

Consider the variational problem

$$J_K[\cdot] \to \max J_K[w] \quad \text{on } \mathcal{C},$$

and recall that if there is some $w_0 \in \mathcal{C}$ such that

$$\sup_{w \in \mathcal{C}} J_K[w] = J_K[w_0],$$

then (2.1) has a solution.

First we observe that $J_K[\cdot]$ is bounded from above. Indeed, by Corollary 2.10, (2.11)

$$\log \fint_{S^2} K e^{2w}\, dv_{g_c} \le \log \max_{S^2} K + \frac{a}{4\pi} \int_{S^2} |\nabla w|^2\, dv_{g_c} + 2 \fint_{S^2} w\, dv_{g_c} + C_4,$$

which leads to

$$J_K[w] \le \log \max_{S^2} K + (a-1) \frac{1}{4\pi} \int_{S^2} |\nabla w|^2\, dv_{g_c} + C_4 < \infty,$$

since $a = \frac{1}{2} < 1$. Taking a maximizing sequence $\{w_l\}_{l \in \mathbb{N}} \subset \mathcal{C}$ with

$$\lim_{l \to \infty} J_K[w_l] = \sup_{w \in \mathcal{C}} J_K[w] =: L$$

we obtain

$$\left(\frac{1-a}{4\pi}\right)\int_{S^2}|\nabla w_l|^2\,dv_{g_c} \le \log\max_{S^2} K + C_4 - J_K[w_l]$$
$$\le \log\max_{S^2} K + C_4 + \varepsilon - L$$

for some $\varepsilon > 0$. This implies by the Poincaré inequality that the w_l are uniformly bounded in $W^{1,2}(S^2)$, hence $w_l \rightharpoonup w_0$ in $W^{1,2}(S^2)$ for some subsequence. Since all w_l are even a.e., clearly w_0 is even a.e. by Rellich's Theorem. Moreover we know that by the definition of $J_K[\cdot]$ in (2.5)

$$\left|\log \fint_{S^2} K e^{2w_l}\,dv_{g_c}\right| \le L + C||w_l||_{W^{1,2}} \le \tilde{C} < \infty,$$

hence

$$\int_{S^2} K e^{2(w_l-\overline{w}_l)}\,dv_{g_c} \ge \min\{4\pi e^{-\tilde{c}}, 1\} =: c_0 > 0. \tag{2.13}$$

This implies by Corollary 1.8 that also

$$\int_{S^2} K e^{2w_0}\,dv_{g_c} \ge c_0 > 0. \tag{2.14}$$

In fact, for $u_l := w_l - \overline{w}_l$, where $\overline{w}_l := \fint_{S^2} w_l\,dv_{g_c}$, and $f := K \in L^\infty(S^2)$, one infers from (1.19)

$$\int_{S^2} K e^{2(w_l-\overline{w}_l)}\,dv_{g_c} \to \int_{S^2} K e^{2(w_0-\overline{w}_0)}\,dv_{g_c},$$

which implies by (2.12), that for any $\varepsilon > 0$, there is $l_0 \in \mathbb{N}$ such that for all $l \ge l_0$

$$(c_0 - \varepsilon)e^{2(\overline{w}_0-\overline{w}_l)} \le \int_{S^2} K e^{2w_0}\,dv_{g_c}.$$

But $w_l \to w_0$ in $L^2(S^2)$ by Rellich's Theorem, hence (2.14) is true. □

Remarks

1. We have omitted the proofs of Theorems 2.4, 2.6, 2.9, due to the limited space. Theorem 2.4 is based on a calculus inequality applied to radially symmetric functions $u = u(|x|)$, to which the problem can be reduced, whereas the proof of Theorem 2.6 is more sophisticated. One reduces the problem to $u = u(x_3)$ working in spherical coordinates. A similar but more complicated reduction is done in the proof of Theorem 2.9.

It should be pointed out that these methods do not carry over to energies with higher order derivatives of u, since the heavily used relation

$$\int_{\mathbb{R}^n} |\nabla u^*|^n\,dx \le \int_\Omega |\nabla u|^n\,dx$$

for the symmetric rearrangement u^* of u, is not valid for higher order energies.

2. For a geometric interpretation[3] of the constants α_n in Theorem 2.4, we look at the following isoperimetric problem for level sets. Let $u \in C^\infty(\Omega)$ be a Morse function.

$$L_t(u) := \text{ length } (\{x \in \Omega : |u(x)| = t\}), \; A_t(u) \quad := \quad \text{area } \{x \in \Omega : |u(x)| \geq t\},$$

then the classical isoperimetric inequality states that

$$\frac{L_t^2}{A_t} \geq 4\pi.$$

Defining $\alpha_2(u) := \lim\inf_{t\to\infty} \frac{L_t^2(u)}{A_t(u)}$ for $u \in W_0^{1,2}(\Omega)$ one obtains

$$\inf_{u \in W_0^{1,2}(\Omega)} \alpha_2(u) = 4\pi,$$

and the infimum is attained for $u \in C^\infty(\Omega)$ with circular level curves.

If $u \in W^{1,2}(\Omega), \Omega \subset \mathbb{R}^2$ with $\int_\Omega u\,dx = 0$, then

$$\frac{L_t^2(u)}{A_t(u)} \geq \begin{cases} 2\pi, \text{ if } \quad \partial\Omega \in C^2, \\ 2\min_i \theta_i, \text{ if } \partial\Omega \text{ is piecewise smooth} \\ \qquad\qquad \text{with interior boundary angle } \theta_i. \end{cases}$$

If $w \in W^{1,2}(S^2)$ with $\int_{S^2} w\,dv_{g_c} = 0, w$ even, then $\alpha_2(w) \geq 8\pi$.

Indeed, the isoperimetric inequality on S^2 for a closed curve with length L and enclosed area A says

$$L^2 \geq A(4\pi - A),$$

which implies

$$\alpha_2(v) = \lim_{t\to\infty} \frac{L_t^2(v)}{A_t(v)} \geq \lim_{t\to\infty}(4\pi - A_t(v)) = 4\pi \tag{2.15}$$

for all $v \in W^{1,2}(S^2)$ with $\int_{S_2} v\,dv_{g_c} = 0$.

This explains the term 4π in the exponential in (2.8) of Theorem 2.6. In particular, for w even, the level curves of w split in two equal parts of length $L_{t,1} = L_{t,2} = L_t/2$. The same holds true for the enclosed areas

$$A_{t,1} = A_{t,2} = A_t/2,$$

which implies

$$\alpha_2(w) = \lim_{t\to\infty} \frac{L_t^2(w)}{A_t(w)} = \lim_{t\to\infty} \frac{4L_{t,1}^2(w)}{2A_{t,1}(w)} \underset{(2.15)}{\geq} 2\cdot 4\pi;$$

compare to Theorem 2.9, where 8π occurs in the exponential in (2.10).

Notice that it is not clear if this geometric interpretation extends to the general case $n \geq 3$ because of the more complicated geometries of level sets.

[3] [32]

We now give a sharpened version of Corollary 2.7, the Onofri inequality.

Theorem 2.11 [Onofri] *Let $w \in W^{1,2}(S^2)$. Then*

$$\log \fint_{S^2} e^{2w}\, dv_{g_c} \le \frac{1}{4\pi} \int_{S^2} |\nabla w|^2\, dv_{g_c} + 2 \fint_{S^2} w\, dv_{g_c}, \tag{2.16}$$

with equality iff

$$\Delta w + e^{2w} = 1, \tag{2.17}$$

i.e.,

$$K_{g_w} \equiv K_{g_c} \equiv 1, \tag{2.18}$$

iff $w = \frac{1}{2}\log|J_\phi|$, where $\phi : S^2 \to S^2$ is a conformal transformation of S^2. In other words, equality in (2.16) *holds iff*

$$e^{2w} g_c = \phi^*(g_c). \tag{2.19}$$

Remark 2.12 An analytic proof for the equivalence of (2.17) and (2.19) was given by Struwe and Uhlenbeck. The equivalence of (2.18) and (2.19) is the content of the classical Cartan–Hadamard Theorem. We will see later when deriving the Polyakov formula, why the Onofri inequality (which sharpens Corollary 2.7, allowing $C_2 = 0$ in (2.9)) is important.

Sketch of the proof of Theorem 2.11. The key idea is a result of Aubin.[5]

Lemma 2.13 [Aubin] *Let $\mathcal{S} := \{w \in W^{1,2}(S^2) : \int_{S^2} e^{2w} x_j\, dv_{g_c} = 0, j = 1, 2, 3\}$. Then for $w \in \mathcal{S}$ the following is true: For all $\varepsilon > 0$ there is a constant C_ε such that*

$$\log \fint_{S^2} e^{2w}\, dv_{g_c} \le \left(\frac{1}{2} + \varepsilon\right) \frac{1}{4\pi} \int_{S^2} |\nabla w|^2\, dv_{g_c} + 2 \fint_{S^2} w\, dv_{g_c} + C_\varepsilon. \tag{2.20}$$

Notice that the *symmetric class* $\mathcal{S}$ is not too special, since for each $w \in C^1(S^2)$ there is a conformal transformation $\phi : S^2 \to S^2$, such that

$$T_\phi(w) := w \circ \phi + \frac{1}{2}\log|J_\phi| \quad \text{is in } \mathcal{S}.$$

In fact T_ϕ gives a $1-1$ correspondence.

Using (2.20) one can obtain compactness for maximizing sequences of $J_K[\cdot]$ on $\mathcal{S}$, see (2.5). The Euler–Lagrange equation for this constrained variational problem contains Lagrange multipliers, that can be shown to vanish using the Kazdan–Warner condition, Theorem 2.1. Finally, the uniqueness of the solution to (2.17), which then is the Euler–Lagrange equation for $J_K[\cdot]$ on $\mathcal{S}$, leads to $w_* \equiv 0$ as the minimizer. (2.16) follows from $0 = J_K[0] = J_K[w_*] \le J_K[w]$ for all $w \in W^{1,2}(S^2)$ (see [72]). $\square$

Remarks

1. For nonsymmetric $K > 0$ Chang and Yang [31], [32] have proved an index formula for (2.1) under very mild nondegeneracy conditions on K, e.g., for Morse functions K, based on the Moser–Trudinger inequality. For general K, K.C. Chang and Liu [21] have extended these results.
2. Solutions of (2.17), or equivalently (2.19), are unique, which is proven by stereographic projection

$$\pi : (S^n - \text{ northpole }) \to \mathbb{R}^n$$
$$\xi \xmapsto{\pi} x(\xi)$$

with inverse $\xi = \pi^{-1}(x),\ \xi_i = \frac{2x_i}{1+|x|^2},\ \xi_{n+1} = \frac{|x|^2-1}{|x|^2+1}$.

For $n = 2$ the transformed equation becomes

$$-\Delta u = e^{2u} \quad \text{on } \mathbb{R}^2, \tag{2.21}$$

where

$$u(x) = \log \frac{2}{1+|x|^2} + w(\xi(x)). \tag{2.22}$$

Assuming $\int_{\mathbb{R}^2} e^{2u}\, dx < \infty$, W.X. Chen and C. Li [36] proved that (2.21) holds iff $u(x) = \log \frac{2\lambda}{\lambda^2+|x-x_0|^2}$, for some $\lambda > 0, x_0 \in \mathbb{R}^2$. Hence $\int_{\mathbb{R}^2} e^{2u(x)} dx = 4\pi = |S^2|$.

Note that without the assumption $\int_{\mathbb{R}^2} e^{2u}\, dx < \infty$, there are actually other analytic solutions to (2.21). In fact, one has a complete picture of the solutions of this equation on $\mathbb{R}^2$, see the classification of [38]. On $\mathbb{R}^n, n \geq 3$, Caffarelli, Gidas and Spruck [16] developed a full theory regarding the equation $-\Delta u = u^{\frac{n+2}{n-2}}$. The idea of projecting equations on S^n to $\mathbb{R}^n$ will also be useful for higher-order problems leading to $(-\Delta)^{n/2} u = (n-1)! e^{nu}$ instead of (2.21).

§ 3 Polyakov formula on compact surfaces

Theorem 3.1 *Suppose* (M^2, g_0) *is a compact surface,* $g_w := e^{2w} g_0$ *is a metric conformal to* g_0*, with* $\mathrm{vol}(M, g_w) = \mathrm{vol}(M, g_0)$.

Then

$$F[w] := \log \frac{\det(-\Delta_{g_w})}{\det(-\Delta_{g_0})} = -\frac{1}{12\pi} \int_M (|\nabla_0 w|^2 + 2K_{g_0} w)\, dv_0. \tag{3.1}$$

On (S^2, g_c) we denote $S[w] := \underset{S^2}{⨍} |\nabla_{g_c} w|^2\, dv_{g_c} + 2 \underset{S^2}{⨍} w\, dv_{g_c}$.

As a consequence of Theorem 3.1 and Onofri's inequality (Theorem 2.11) we obtain

Corollary 3.2 *On* (S^2, g_c)*, one has*

$$\log \frac{\det(-\Delta_{g_w})}{\det(-\Delta_{g_c})} = -\frac{1}{3} S[w] \le 0 \tag{3.2}$$

for all $w \in C^\infty(S^2)$ *with* $\mathrm{vol}(M, g_w) = 4\pi$*, hence* $F[w] \le F[0]$*, i.e.,* $F[\cdot]$ *is maximal at the standard metric* g_c*, which corresponds to* $w = 0$.

Notice that $\log(\det -\Delta_{g_w})$ is defined via the regularized zeta function as in Ray and Singer ([79]).

Corollary 3.3 *On any compact surface* (M^2, g_0) *with* $K_{g_0} \equiv \mathrm{const.} \le 0$ *and with* $\mathrm{vol}(M, g_0) = 1$ *one has: If* $w \in C^\infty(M)$ *satisfies* $\int_M e^{2w}\, dv_0 = \mathrm{vol}(M, g_w) = 1$*, then*

$$F[w] \le 0$$

with equality only at the constant curvature metric g_0.

Proof. First notice that by Jensen's inequality

$$e^{2\overline{w}} \le ⨍_M e^{2w}\, dv_0 = \int_M e^{2w}\, dv_0 = 1,$$

thus $\overline{w} \le 0$, where $\overline{w} := \underset{M}{⨍} w dv_0 = \int_M w\, dv_0$. $K_{g_0} \le 0$ implies $\int_M 2K_{g_0} w\, dv_0 = 2K_{g_0} \int_M w\, dv_0 \ge 0$, hence $F[w] \le 0$. □

Observe that the above argument leads to

$$\int_M |\nabla_0 w|^2\, dv_0 \le -12\pi F[w],$$

which means that spectral information given by $F[w]$ bounds the energy of w. For a related result in case of the sphere ($K_{g_0} = K_{g_c} \equiv 1$) we refer to the end of this section for a result by Osgood–Phillips–Sarnak.

For the definition of the *zeta functional determinant* $\log(\det -\Delta_g)$, we consider a compact Riemannian manifold (M^n, g), $\partial M = \emptyset$ with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \tag{3.3}$$

denoting the eigenvalues of the Laplace–Beltrami operator

$$-\Delta_g := -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{g}\frac{\partial}{\partial x_j}\right), \sqrt{g} := \sqrt{\det g},\ g^{ij} := (g_{ij})^{-1}. \tag{3.4}$$

The eigenfunctions $\{\phi_j\}$ form an orthonormal basis for $L^2(M)$ and satisfy

$$\Delta_g \phi_j + \lambda_j \phi_j = 0 \quad \text{on } M. \tag{3.5}$$

We consider the zeta function

$$\zeta(s) := \sum_{\lambda_k \neq 0} \lambda_k^{-s}, \tag{3.6}$$

and observe that formal differentiation leads to

$$\zeta'(s) = \sum_{\lambda_k \neq 0} -(\log \lambda_k)\lambda_k^{-s}, \text{ i.e.,}$$

$$\zeta'(0) = -\sum_{\lambda_k \neq 0} \log \lambda_k = -\log \prod_{k=1}^{\infty} \lambda_k.$$

This formal computation motivates the definition of the log-determinant according to Ray and Singer [79] as

$$\log\det(-\Delta_g) := -\zeta'(0). \tag{3.7}$$

We will now justify the existence of $\zeta'(0)$. Denote $N(\lambda) := \#\{j \in \mathbb{N} : \lambda_j \leq \lambda\}$ as the counting function and recall Weyl's asymptotic formula:

Proposition 3.4 *Let (M^n, g) be compact with $\partial M = \emptyset$. Then*

$$N(\lambda) \sim \omega_n \operatorname{vol}(M, g)\frac{\lambda^{n/2}}{(2\pi)^n}, \textit{ as } \lambda \to \infty, \tag{3.8}$$

i.e.,

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n}\operatorname{vol}(M, g), \tag{3.9}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. In particular, for $\lambda = \lambda_k$,

$$(\lambda_k)^{\frac{n}{2}} \sim \frac{k \cdot (2\pi)^n}{w_n \operatorname{vol}(M, g)}, \textit{ as } k \to \infty, \tag{3.10}$$

i.e., λ_k grows like $k^{\frac{2}{n}}$ as k tends to ∞.

The asymptotic relation (3.10) implies that $\zeta(s)$ is well defined for $Re\,(s) > \frac{n}{2}$. To justify the expression $\zeta'(0)$ in (3.7) recall the *Mellin transform*

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt} t^{s-1}\, dt, \tag{3.11}$$

where $\Gamma(s)$ denotes the value of the Gamma function at s:

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\, dt.$$

Note that $\Gamma(s)$ has a simple pole at $s = 0$,

$$\lim_{s \to 0} \Gamma(s) s = 1. \tag{3.12}$$

Using (3.11) we can rewrite $\zeta(s)$ in terms of the Gamma function for $Re\,(s) > \frac{n}{2}$:

$$\begin{aligned} \zeta(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{j=1}^\infty e^{-\lambda_j t} t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty (Z(t) - 1) t^{s-1}\, dt, \end{aligned}$$

where

$$Z(t) := \int_M H(x, x, t)\, dv_g(x) = \sum_{k=0}^\infty e^{-\lambda_k t} = Tr(e^{t\Delta_g}) \tag{3.13}$$

is the trace of the heat kernel

$$H(x, y, t) := \sum_{k=0}^\infty e^{-\lambda_k t} \phi_k(x) \phi_k(y). \tag{3.14}$$

Proposition 3.5 [67], [66] *$H(x, y, t)$ is the unique fundamental solution of the heat equation*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_g u = 0, \\ \lim_{t \to 0} u(x, t) = f(x), \end{cases} \tag{3.15}$$

on M^n (M compact, closed), i.e., for any given $f \in C^\infty(M)$, the convolution $u := H * f$ solves (3.15). *Moreover H is continuous on $M \times M \times (0, \infty)$, and $H(\cdot, \cdot, t) \in C^2(M \times M), H(x, y, \cdot) \in C^1((0, \infty))$. In addition*[4],

$$H(x, x, t) \sim \left(\frac{1}{4\pi}\right)^{\frac{n}{2}} \sum_{k=0}^\infty B_k(x) t^{\frac{k-n}{2}}, \ \textit{as}\ t \to 0^+, \tag{3.16}$$

where B_k are local invariants of M of order k. $B_k \equiv 0$ for all odd k, ($\partial M = \emptyset$).

[4]Definition: $A(t) \sim B(t)$ iff $\lim_{t \to 0} \frac{A(t) - B(t)}{t^m} = 0$ for all $m \geq 0$.

Consequently, by (3.13) and (3.16)

$$Z(t) \sim \left(\frac{1}{4\pi}\right)^{\frac{n}{2}} \sum_{k=0}^{\infty} a_k t^{\frac{k-n}{2}}, \text{ as } t \to 0^+, \tag{3.17}$$

where $a_k := a_k(\Delta_g) := \int_M B_k(x)\, dv_g(x)$ are the *heat coefficients* of M.

For $n = 2$, (3.16) and (3.17) can be computed as

$$H(x,x,t) = \frac{1}{4\pi t} + \frac{K(x)}{12\pi} + \frac{K^2(x)t}{60\pi} + O(t^2), \text{ as } t \to 0^+, \tag{3.18}$$

$$Z(t) = \frac{\mathrm{vol}(M,g)}{4\pi t} + \frac{\chi(M)}{6} + \frac{\pi t}{60}\int_M K^2\, dv_g + O(t^2), \text{ as } t \to 0^+. \tag{3.19}$$

In particular, $a_0 = \mathrm{vol}(M,g)$, $a_2 = \frac{1}{3}\int_M K\, dv_g = \frac{2\pi}{3}\chi(M)$.

Thus, wherever the zeta function converges, we have

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^1 (Z(t)-1)t^{s-1}\, dt + \frac{1}{\Gamma(s)}\int_1^{\infty} (Z(t)-1)t^{s-1}\, dt \\
&= \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left[\frac{\mathrm{vol}(M,g)}{4\pi t} + \frac{\chi(M)}{6} + \frac{\pi t}{60}\int_M K^2\, dv_g + t^2 P(t) - 1\right] dt \\
&\quad + \frac{1}{\Gamma(s)}\int_1^{\infty}\left(\sum_{k=1}^{\infty} e^{-\lambda_k t}\right) t^{s-1}\, dt,
\end{aligned}$$

where $P(t)$ is a bounded function in t. The second integral is holomorphic in s, since $\Gamma(s)$ does not vanish, and since $\sum_{k=1}^{\infty} e^{-\lambda_k t} \le Ce^{-\lambda_1 t}$ for large t, by (3.10).

The first integral may be written as

$$\frac{1}{\Gamma(s)}\left[\frac{t^{s-1}}{s-1}\cdot\frac{\mathrm{vol}(M,g)}{4\pi} + \frac{\chi(M)}{6s}t^s + \frac{\pi t^{s+1}}{60(s+1)}\int_M K^2 dv_g - \frac{t^s}{s}\right]_{t=0}^{t=1} + B(s),$$

where $B(s) = \frac{1}{\Gamma(s)}\int_0^1 t^{s+1}P(t)\, dt$ is holomorphic for $Res > -1$. The above expression converges for all $s \in \mathbb{C}$ with $Re(s) > 1$, and has a meromorphic continuation to all of $\mathbb{C}$ with a simple pole at $s = 1$.

To summarize these observations, $\zeta(s)$ is holomorphic for $Re(s) > 1$, has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ and with

$$\zeta(0) = \frac{\chi(M)}{6} - 1. \tag{3.20}$$

(See, e.g., Rosenberg [81], Chapter 5, for the corresponding result for general $n \ge 2$.)

Hence $\zeta(s)$ is analytic at $s = 0$, which means that

$$\zeta'(0) := \lim_{s\to 0}\frac{\zeta(s)-\zeta(0)}{s}$$

exists, and (3.7) is justified.

Remark 3.6 The notion of log-determinant of the Laplacian was introduced in [79] to define *analytic torsion* T by

$$\log T := \frac{1}{2}\sum_{q=0}^{n}(-1)^q q\zeta_q'(0),$$

where

$$-\zeta_q'(0) := \log\det(-\Delta_q),$$

Δ_q = Laplacian on q-forms. Cheeger [35] and W. Müller [68] proved independently later that this notion of analytic torsion coincides with a topological quantity, namely the *Reidemeister torsion.*

To prove Theorem 3.1 we need to look at a more general version of Proposition 3.5, as defined by Branson and Gilkey. ([13])

Proposition 3.7 (Branson–Gilkey) *Let $\varphi \in C^\infty(M), (M^n, g)$ closed and compact, and set $H_\varphi(x,t) := \varphi(x)H(x,x,t)$,*

$$Z_\varphi(t) := Tr(\varphi e^{\Delta_g t}) = \int_M H_\varphi(x,t)\, dv_g(x)$$

with $H(x,y,t)$ as in (3.14).

Then there are coefficients $B_k(\varphi,\Delta_g)(\cdot), a_k(\varphi,\Delta_g)$, such that $B_k(\varphi,\Delta_g)\equiv 0$ for k odd,

$$H_\varphi(x,t) \sim \left(\frac{1}{4\pi}\right)^{\frac{n}{2}} \sum_{k=0}^{\infty} B_k(\varphi,\Delta_g)_{(x)} t^{\frac{k-n}{2}},\ \text{as } t\to 0^+, \tag{3.21}$$

$$Z_\varphi(t) \sim \left(\frac{1}{4\pi}\right)^{\frac{n}{2}} \sum_{k=0}^{\infty} a_k(\varphi,\Delta_g) t^{\frac{k-n}{2}},\ \text{as } t\to 0^+ \tag{3.22}$$

with $B_k(\varphi,\Delta_g)(x) = \varphi(x)B_k(x), B_k(x)$ as in (3.16), *and*

$$a_k(\varphi,\Delta_g) = \int_M \varphi(x)B_k(x)\, dv_g(x). \tag{3.23}$$

(In particular, $a_k = 0$ for k odd.)

Notice that with this notation $a_k(1,\Delta_g) = a_k(\Delta_g) = a_k$ as defined in (3.17), in particular

$$a_0(\varphi,\Delta_g) = \int_M \varphi(x)\, dv_g(x), \tag{3.24}$$

$$a_2(\varphi,\Delta_g) = \frac{1}{3}\int_M \varphi(x)K_g(x)\, dv_g(x) \tag{3.25}$$

Proof of Theorem 3.1. The following lemma is the crucial step in the proof of Theorem 3.1.

Lemma 3.8 (Key Lemma) *Suppose (M^2, g_0) is closed and compact. Then*

$$\frac{d}{d\varepsilon}_{|\varepsilon=0} \zeta'_{\Delta_{u+\varepsilon\varphi}}(0) = \frac{a_2(\varphi, \Delta_u)}{2\pi} - 2\frac{\int_M \varphi dv_{g_u}}{\int_M dv_{g_u}}, \tag{3.26}$$

where we have set $\Delta_u := \Delta_{g_u}, g_u := e^{2u}g_0$.

We defer the proof of this lemma to the end of this chapter and apply (3.26) to prove Theorem 3.1 first:

By (3.7) we obtain

$$\begin{aligned}
-\log\frac{\det(-\Delta_{g_w})}{\det(-\Delta_{g_0})} &= \zeta'_{\Delta_w}(0) - \zeta'_{\Delta_0}(0) = \int_0^1 \frac{d}{dt}(\zeta'_{\Delta_{tw}}(0))\,dt \\
&= \int_0^1 \frac{a_2(w, \Delta_{tw})}{2\pi}\,dt - 2\int_0^1 \frac{\int_M we^{2tw}dv_0}{\int_m e^{2tw}dv_0}dt \\
&\underset{(3.25)}{=} \frac{1}{6\pi}\int_0^1 \left(\int_M wK_{g_{tw}}\,dv_{g_{tw}}\right)dt - (\log\frac{\int_M e^{2w}dv_0}{\int_M dv_0}) \\
&\underset{(1.3)}{=} \frac{1}{6\pi}\int_0^1 \left(\int_M w(-\Delta_0(tw) + K_{g_0})\,dv_0\right)\,dt \\
&= \frac{1}{6\pi}\int_0^1 \left(t\int_M |\nabla_0 w|^2\,dv_0 + \int_M K_{g_0}w\,dv_0\right)\,dt \\
&= \frac{1}{12\pi}\int_M (|\nabla_0 w|^2 + 2K_{g_0}w)\,dv_0.
\end{aligned}$$

(Notice that we used the identity $\frac{d}{d\varepsilon}_{|\varepsilon=0}\zeta'_{\Delta_{tw+\varepsilon w}}(0) = \frac{d}{dt}\zeta'_{\Delta_{tw}}(0)$ to apply (3.26).) Thus (3.1) is proved. □

Proof of Lemma 3.8. Without justification of every step below we calculate formally:

$$\begin{aligned}
&\frac{d}{d\varepsilon}_{|\varepsilon=0} Tr(e^{t\Delta_{u+\varepsilon\varphi}}) = \frac{d}{d\varepsilon}_{|\varepsilon=0} Tr(e^{te^{-2\varepsilon\varphi}\Delta_u}) \\
&= 2t\cdot Tr(\varphi\Delta_u e^{t\Delta_u}) = -2t\frac{d}{dt}\tilde{T}r(\varphi e^{t\Delta_u}),
\end{aligned} \tag{3.27}$$

where

$$\tilde{T}r(\varphi e^{t\Delta_u}) = Tr(\varphi e^{t\Delta_u}) - \frac{\int_M \varphi dv_{g_u}}{\int_M dv_{g_u}},$$

and where we used that $\Delta_{g_w} = e^{-2w}\Delta_g$ for $n = 2$, as can easily be checked by (3.4).

Therefore, formally,

$$\begin{aligned}
&\frac{d}{d\varepsilon}_{|\varepsilon=0}\frac{d}{ds}_{|s=0}\zeta_{\Delta_{u+\varepsilon\varphi}}(s)=\frac{d}{ds}_{|s=0}\frac{d}{d\varepsilon}_{|\varepsilon=0}\zeta_{\Delta_{u+\varepsilon\varphi}}(s)\\
&\underset{(3.13)}{=}\frac{d}{ds}_{|s=0}\frac{d}{d\varepsilon}_{|\varepsilon=0}\frac{1}{\Gamma(s)}\int_0^\infty (Tr(e^{t\Delta_{u+\varepsilon\varphi}})-1)\,t^{s-1}\,dt\\
&=\frac{d}{ds}_{|s=0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{d}{d\varepsilon}_{|\varepsilon=0}Tr(e^{t\Delta_{u+\varepsilon\varphi}})\,dt\\
&\underset{(3.27)}{=}\frac{d}{ds}_{|s=0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(-2t\frac{d}{dt}\tilde{T}r(\varphi e^{t\Delta_u})\right)dt\\
&=\frac{d}{ds}_{|s=0}\frac{1}{\Gamma(s)}\left\{\left[-2t^s\tilde{T}r(\varphi e^{t\Delta_u})\right]_{t=0}^{t=\infty}+2\int_0^\infty st^{s-1}\tilde{T}r(\varphi e^{t\Delta_u})\,dt\right\}\\
&=2\frac{d}{ds}_{|s=0}\left\{\frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\tilde{T}r(\varphi e^{t\Delta_u})\,dt\right\}\\
&=2\frac{d}{ds}_{|s=0}\left(\frac{s}{\Gamma(s)}\left\{\int_0^1 t^{s-1}\tilde{T}r(\varphi e^{t\Delta_u})\,dt+\int_1^\infty t^{s-1}\tilde{T}r(\varphi e^{t\Delta_u})\,dt\right\}\right).
\end{aligned}$$

Notice that there are no boundary terms in the integration by parts, as the integrand is of exponential decay at infinity, and, by the asymptotic behavior near zero (3.21), the integrand vanishes at zero, if $Re(s)$ is sufficiently large.

The last integral is holomorphic in s. In addition, $\Gamma(s)=\frac{1}{s}-\frac{1}{s+1}+\cdots$, hence

$$\frac{s}{\Gamma(s)}=s^2-\frac{s^2}{s+1}+\cdots,\ \text{in particular}\ \frac{d}{ds}_{|s=0}\left(\frac{s}{\Gamma(s)}\right)=0. \tag{3.28}$$

So the only term we need to consider is

$$\begin{aligned}
&\frac{d}{d\varepsilon}_{|\varepsilon=0}\frac{d}{ds}_{|s=0}\zeta_{\Delta_{u+\varepsilon\varphi}}(s)\\
&\underset{(3.22)}{=}2\frac{d}{ds}_{|s=0}\left(\frac{s^2}{4\pi}\int_0^1\sum_{k=0}^\infty a_k(\varphi,\Delta_u)t^{\frac{k-2}{2}+s-1}\,dt-s\fint_M\varphi dv_{g_u}\right)\\
&=\frac{1}{2\pi}\frac{d}{ds}_{|s=0}s^2\Bigg[\frac{a_0(\varphi,\Delta_u)}{s-1}t^{s-1}\\
&\quad+\frac{a_2(\varphi,\Delta_u)}{s}t^s+2\sum_{k=4}^\infty\frac{a_k(\varphi,\Delta_u)}{k+2s-2}t^{\frac{k+2s-2}{2}}\Bigg]_{t=0}^{t=1}-2\fint_M\varphi dv_{g_u}\\
&=\frac{1}{2\pi}\frac{d}{ds}_{|s=0}\left\{\frac{s^2}{s-1}a_0(\varphi,\Delta_u)+sa_2(\varphi,\Delta_u)+2s^2\sum_{k=4}^\infty\frac{a_k(\varphi,\Delta_u)}{k+2s-2}\right\}\fint_M\varphi dv_{g_u}\\
&=\frac{a_2(\varphi,\Delta_u)}{2\pi}-\fint_M\varphi dv_{g_u},
\end{aligned}$$

which proves (3.26). $\square$

Theorem 3.9 (Osgood–Phillips–Sarnak [73], [74]) *Isospectral metrics on a closed compact surface (M^2, g) are C^∞-compact modulo the isometry class.*

The basic idea in the proof is that on a compact closed surface (M^2, g_0), each heat coefficient a_{2i} for each $i \geq 2$ controls the Sobolev $W^{i,2}$-norm modulo some lower order $W^{l,2}$-norm for $l < i$ of the conformal factor w for the metric $g_w = e^{2w} g_0$. But when $i = 1$, $a_2 = \frac{2\pi}{3}\chi(M)$ is only a (topological) constant. Thus to control the $W^{1,2}$-norm of w, one needs to replace a_2 by some other isospectral information – which is provided by the log determinant functional $F[w]$ as defined in (3.1).

Sketch of the proof. Without loss of generality one can choose the background metric g_0 such that $K_{g_0} \equiv -1, 0$, or $+1$. For a sequence of isospectral metrics $g_{w_k}, a_0 = \mathrm{vol}(M, g_{w_k})$ is fixed. Moreover, by (3.1)

$$F_0 \equiv F[w_k] = -\frac{1}{12\pi} \int_M (|\nabla_0 w_k|^2 + 2K_{g_0} w_k)\, dv_0.$$

If $K_{g_0} = 0$ or $K_{g_0} = -1$ we get a uniform $W^{1,2}$-bound on w_k by the observation after Corollary 3.3 and Trudinger's embedding theorem (Corollary 1.7 in §1).

For $K_{g_0} = 1$ one uses conformal transformations $\phi : S^2 \to S^2$ and Aubin's Lemma (Lemma 2.13) as in the proof of Onofri's inequality, Theorem 2.11, to work in the symmetric class $\mathcal{S}$. Then one obtains a uniform bound on

$$\int_{S^2} |\nabla_{g_c}(T_\phi(w_k))|^2\, dv_{g_c} \quad \text{in terms of} \quad F[T_\phi(w_k)] = F[w_k]$$

because of the isometric invariance of the spectrum. This together with the fact that the volume of the metric $g_{T_\phi(w_k)}$ is always a_0 leads to a uniform bound on $||w_k||_{1,2}$. The higher-order coefficients a_{2i} then enable us to control the $W^{i,2}$-norms of w as well, for all $i \in \mathbb{N}$. $\square$

§ 4 Conformal covariant operators – Paneitz operator

Let (M^n, g_0) be a compact n-dimensional manifold with $\partial M = \emptyset$. We consider a formally selfadjoint geometric differential operator, i.e., an operator defined in terms of geometric quantitives of (M, g_0). We say that A is *conformally covariant of bidegree* (a, b) iff

$$A_{g_w}(\varphi) = e^{-bw} A_{g_0}(e^{aw}\varphi) \quad \text{for all } \varphi \in C^\infty(M). \tag{4.1}$$

Examples

1. The Laplace–Beltrami operator for $n = 2$,

$$\Delta_g := \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_j} \right),$$

satisfies

$$\Delta_{g_w} = e^{-2w} \Delta_{g_0}, \text{ i.e.,} \tag{4.2}$$

Δ_{g_0} is conformally covariant of bidegree $(a, b) = (0, 2)$. Recall that in this case

$$\Delta_0 w + K_{g_w} e^{2w} = K_{g_0}, \tag{4.3}$$

which is the *Gaussian curvature equation.*

2. The conformal Laplacian for $n \geq 3$,

$$L_g := -\Delta_g + \frac{n-2}{4(n-1)} R_g,$$

satisfies

$$L_{g_w}(\varphi) = e^{-\frac{n+2}{2}w} L_{g_0} \left(e^{\frac{n-2}{2}w} \varphi \right) \quad \text{for all } \varphi \in C^\infty(M), \tag{4.4}$$

hence L_g is conformally covariant of bidegree $\left(\frac{n-2}{2}, \frac{n+2}{2}\right)$.

Notice that $b - a = 2$ in Examples 1 and 2. The usual notation $g_u := u^{\frac{4}{n-2}} g_0 := e^{2w} g_0$ leads to

$$L_{g_u}(\varphi) = u^{-\frac{n+2}{n-2}} L_{g_0}(u\varphi) \quad \text{for all } \varphi \in C^\infty(M) \tag{4.5}$$

instead of (4.4). In particular, for $\varphi \equiv 1$,

$$L_{g_u}(1) = u^{-\frac{n+2}{n-2}} L_{g_0}(u), \tag{4.6}$$

and more explicitly,

$$-\Delta_0 u + c_n R_{g_0} u = c_n u^{\frac{n+2}{n-2}} R_{g_u}, \tag{4.7}$$

where $c_n := \frac{n-2}{4(n-1)}$, which is the *scalar curvature equation* or *Yamabe equation.*

Here we will present a formal argument to derive (4.3) from (4.4) which we learned from Tom Branson. The argument runs as follows: with a formal limit

$n \searrow 2$ after analytic continuation one finds that (4.3) appears as a special case of (4.4): Taking $\varphi \equiv 1$ in (4.4) we get

$$\begin{aligned}\left(-\Delta_0 + \frac{n-2}{4(n-1)}R_{g_0}\right)\left(e^{\frac{n-2}{2}w}\right) &\underset{(4.4)}{=} e^{\frac{n+2}{2}w}\left(-\Delta_{g_w} + \frac{n-2}{4(n-1)}R_{g_w}\right)(1)\\ &= e^{\frac{n+2}{2}w}\frac{n-2}{4(n-1)}R_{g_w}.\end{aligned}$$

Adding $0 = \Delta_0(1)$ on the left-hand side leads to

$$-\Delta_0\left(e^{\frac{n-2}{2}w} - 1\right) + c_n R_{g_0} e^{\frac{n-2}{2}w} = e^{\frac{n+2}{2}w} c_n R_{g_w}.$$

Dividing both sides by $\frac{n-2}{2}$ and taking the formal limit $n \searrow 2$ we arrive at

$$\begin{aligned}&-\Delta_0\left(\frac{2}{n-2}\left(e^{\frac{n-2}{2}w} - 1\right)\right) + \frac{1}{2(n-1)}R_{g_0}e^{\frac{n-2}{2}w} = e^{\frac{n+2}{2}w}\frac{1}{2(n-1)}R_{g_w},\\ \Rightarrow &-\Delta_0 w + \frac{R_{g_0}}{2} = e^{2w}\frac{R_{g_w}}{2},\end{aligned}$$

which is (4.3), since $R_{g_0} = 2K_{g_0}, R_{g_w} = 2K_{g_w}$, and

$$\text{“}\lim_{n\to 2}\frac{2}{n-2}\left(e^{\frac{n-2}{2}w} - 1\right) = \lim_{a\to 0}\frac{e^{aw} - e^{0\cdot w}}{a-0} = \frac{d}{da}e^{aw}|_{a=0} = w\text{”}.$$

3. The first higher-order example of conformally covariant operators for $n = 4$ is the Paneitz operator [75] given by

$$P_4 := (-\Delta_g)^2 - \operatorname{div}_g\left(\frac{2}{3}R_g g_{ij} - 2R_{ij}\right)d, \tag{4.8}$$

where d is the differential (acting on functions). If we denote by δ the negative divergence, we can rewrite (4.8) as

$$(P_4)_g = (-\Delta_g)^2 + \delta\left(\frac{2}{3}R_g g_{ij} - 2R_{ij}\right)d. \tag{4.9}$$

This leads to

$$\begin{aligned}\langle (P_4)_g\varphi, \psi\rangle_{L^2(dv_g)} = &\int_M (\Delta_g \cdot \Delta_g\varphi)\psi\, dv_g + \int_M \frac{2}{3}R_g\langle\nabla_g\varphi, \nabla_g\psi\rangle_g\, dv_g\\ &-2\int_M \operatorname{Ric}(\nabla_g\varphi, \nabla_g\psi)\, dv_g.\end{aligned}$$

The Paneitz operator P_4 has the basic properties

$$(P_4)_{g_w} = e^{-4w}(P_4)_{g_0}, \quad \text{i.e.,} \tag{4.10}$$

$(P_4)_g$ is conformally covariant with degree $(0, 4)$.

Moreover,

$$(P_4)_{g_0} w + 2Q_{g_0} = 2Q_{g_w} e^{4w}, \tag{4.11}$$

where

$$12Q_g := R_g^2 - 3|\mathrm{Ric}_g|_g^2 - \Delta_g R_g, \tag{4.12}$$

with $|\cdot|_g$ being the Hilbert–Schmidt norm, with respect to the metric g, i.e.,

$$|\mathrm{Ric}_g|_g^2 := \sum_{i,j=1}^{n} |(R_{ij})_g|_g^2.$$

Rewriting (4.11) as $-(P_4)_{g_0} w + 2Q_{g_w} e^{4w} = 2Q_{g_0}$ we discover the similarity to (4.3), and we can interpret Δ_g as $-(P_2)_g$.

In general, it is tedious to check formulas (4.10) and (4.11).

We will here consider two simple examples of the Paneitz operator.

3a. On $\mathbb{R}^4$ (or $\Omega \subset \mathbb{R}^4$) with the flat metric $g = |dx|^2$ we have $R = 0, R_{ij} = 0$ and the Paneitz operator reduces to

$$(P_4)_g = (-\Delta_g)^2. \tag{4.13}$$

3b. If (M^4, g_c) is an Einstein manifold, i.e., with $(R_{ij})_{g_c} = \frac{1}{4} R_{g_c} (g_c)_{ij}$, $R_{g_c} \equiv$ const. for the canonical metric g_c, we get

$$\begin{aligned}(P_4)_{g_c} &= (-\Delta_{g_c})^2 - \frac{1}{6} R_{g_c} \Delta_{g_c} \\ &= (-\Delta_{g_c}) \left(-\Delta_{g_c} + \frac{1}{6} R_{g_c} \right) \\ &= (-\Delta_{g_c}) \circ L_{g_c},\end{aligned} \tag{4.14}$$

where L_{g_c} is the conformal Laplacian discussed as Example 2. (4.14) holds true, since $\delta d = - * d * d = -\Delta$.

3c. As a special example we take (S^4, g_c) with $R_{g_c} \equiv 12$, then (4.14) reads as

$$(P_4)_{g_c} = (-\Delta_{g_c}) \circ (-\Delta_{g_c} + 2). \tag{4.15}$$

4. In the same paper [75], Paneitz also introduced the *conformal Paneitz operators* $(P_4^n)_g$. Setting

$$\begin{aligned}J_g &:= \frac{R_g}{2(n-1)}, \\ A_g &:= (A_{ij})_g := (R_{ij})_g - J_g g_{ij}, \\ (C_{ij})_g &:= \frac{1}{n-2} (A_{ij})_g, \\ (T_g)_{ij} &= (n-2) J_g g_{ij} - 4 C_g g_{ij},\end{aligned}$$

and

$$(Q_4^n)_g := \frac{n}{2} J_g^2 - 2|A_g|^2 - \Delta_g J_g,$$

one gets the operator

$$(P_4^n)_g = (-\Delta_g)^2 + \delta T_g d + \frac{n-4}{2} (Q_4^n)_g, \tag{4.16}$$

and the claim is

$$(P_4^n)_g \text{ is conformally covariant of bidegree } \left(\frac{n-4}{2}, \frac{n+4}{2} \right). \tag{4.17}$$

If one accepts (4.17) one can derive (4.11) from (4.17) in the same way (taking the formal limit $n \searrow 4$) as we deduced (4.3) from (4.4).

Remarks

1. Although the operator P_4^n was introduced by Paneitz, the specific expression of the Q_4^n was introduced by T. Branson [9]. More significantly, in the special case when $n = 4$, Branson has pointed out that Q_4^4 is part of the integrand in the Gauss–Bonnet formula. As we will see in the theorem below, the existence of P_k^n for $k \geq 4$ was established in [49], In ([10], [11]) Branson has also introduced the corresponding Q_k^n-curvatures. We now call these curvatures Q curvatures.
2. Notice in the definition of $(Q_4^n)_g$, that $(Q_4^4)_g = 2Q_g$, compare to (4.12) in Example 3.
3. The tensor A is called the *Weyl–Schouten tensor*; we will discuss some eigenvalue problems of the tensor in later chapters of this lecture notes.

Theorem 4.1 [49] *Let k be a positive even integer. Suppose n is odd, or $k \leq n$.*

Then there is a conformally covariant differential operator P_k on scalar functions of bidegree $\left(\frac{n-k}{2}, \frac{n+k}{2}\right)$ with:

(i) *the leading symbol of P_k is $(-\Delta)^{k/2}$, and on $(\mathbb{R}^n, |dx|^2)$ we have $P_k \equiv (-\Delta)^{k/2}$,*

(ii) *$P_k = P_k^0 + \frac{n-k}{2} Q_k$, where Q_k is a local scalar invariant, and $P_k^0 = \delta S_{k-2} d$. Here, S_{k-2} is a differential operator on 1-forms,*

(iii) *P_k is self-adjoint.*

Remarks

1. This theorem does not assert uniqueness of the operator P_k. For example, one can add $|W|^2$ for $n = 4 : (P_4)_g + |W|_g^2$ has the same properties of the theorem as $(P_4)_g$, where W is the Weyl tensor, which satisfies a pointwise conformal invariant property: $|W|_{g_w}^2 = e^{-4w} |W|_{g_0}^2$.
2. The condition $k \leq n$ is necessary if n is even.

3. The work of [49] is based on the work of Fefferman–Graham, [43], where they regard (M^n, g) as the conformal infinity of (X^{n+1}, g^+) for some asymptotically conformally compact Einstein manifold X^{n+1} satisfying $Ric_{g^+} = -ng^+$. There is a correspondence between the conformal invariants of (M^n, g) and the metric invariants of (X^{n+1}, g^+).
4. Powers of conformally covariant operators are in general not conformally covariant anymore, which can be seen by looking at powers of the conformal Laplacian.

Corollary 4.2 *If n is even, then there exists a curvature metric invariant $(Q_n)_g$ with*

$$\int_M (Q_n)_{g_w} \, dv_{g_w} = \int_M (Q_n)_{g_0} \, dv_0, \tag{4.18}$$

i.e., $\int_M (Q_n)_g \, dv_g$ is a conformally invariant quantity.

Note that $(Q_n)_g = Q_g$ for $n = 4$, see (4.12). For $n = 2$, the total curvature $\int_{M^2} K_g \, dv_g$ satisfies the invariance property (4.18), it is in fact a topological invariant according to the Gauss–Bonnet Theorem.

Proof of Corollary 4.2. Since $(P_n)_{g_0} w + (Q_n)_{g_0} = (Q_n)_{g_w} e^{nw}$ obtained by analytic continuation from the conformal invariance relation for P_n, similar to the case $n = 2, 4$, we can apply part (ii) of Theorem 4.1 [49] for $k = n$. P_n^0 is of the form $\delta S_{n-2} d$, which vanishes after integration. So

$$\int_M (Q_n)_{g_0} \, dv_0 = \int_M (Q_n)_{g_w} e^{nw} \, dv_0 = \int_M (Q_n)_{g_w} \, dv_{g_w}. \qquad \square$$

§ 5 Functional determinant on 4-manifolds

Let (M^n, g) be a compact n-dimensional manifold without boundary and suppose that A is a self-adjoint, geometric differential operator with positive leading symbol of order $2l$. In addition, assume that A scales like its leading symbol, i.e., if $\bar{g} := c^2 g$ for some $c > 0$, then $\bar{A} = (A)_{\bar{g}} = c^{-2l}(A)_g = c^{-2l}A$.

Take, e.g., A as the conformal Laplacian, that is

$$A := L = -\Delta_g + c_n R_g,$$

and compare with Example 2 of Chapter 4.

Then we have the heat kernel expansion with asymptotic behavior

$$Tr(\varphi e^{-tA}) \sim \sum_{k=0}^{\infty} t^{\frac{k-n}{2l}} a_k(\varphi, A), \text{ as } t \to 0^+ \tag{5.1}$$

where

$$a_k(\varphi, A) := \int_M \varphi(x) B_k(x, A)\, dv_g(x)$$

for $\varphi \in C^\infty(M)$, where B_k is a local invariant (in metric g) of order k; compare with Proposition 3.7. Denoting the eigenvalues of A by $\lambda_j, j = 0, 1, 2, \ldots,$ then only finitely many of the λ_j's are negative, since M is compact, and the asymptotic behavior for j tending to ∞ is given by Weyl's formula

$$\lambda_j \sim c(g, A) j^{\frac{2l}{n}},$$

(compare with (3.10) for $A = \Delta_g, l = 1$.)

In analogy to (3.6) the zeta function ζ_A for the operator A is defined as

$$\zeta_A(s) := \sum_{\lambda_j \neq 0} |\lambda_j|^{-s} \text{ for } Re\,(s) > \frac{n}{2l}. \tag{5.2}$$

ζ_A has a meromorphic continuation onto all of $\mathbb{C}$ with simple poles, and is analytic at $s = 0$, which may be proved in a fashion similar to the argumentation used in Chapter 3.

The determinant of A is defined as

$$\det A := (-1)^{\#\{j : \lambda_j < 0\}} \exp(-\zeta_A'(0)), \tag{5.3}$$

hence $|\det A| = \exp(-\zeta_A'(0))$, generalizing (3.7).

Notice that this definition of the determinant is not scaling invariant, that is, for $\bar{g} = c^2 g$, for $c > 0$, one gets $\bar{A} = c^{-2l}A, \bar{\lambda}_j = c^{-2l}\lambda_j$ and

$$\zeta_{\bar{A}}(s) = \sum_{\bar{\lambda}_j \neq 0} |\bar{\lambda}_j|^{-s} = c^{2ls}\zeta_A(s).$$

Hence, although $\zeta_{\bar{A}}(0) = \zeta_A(0)$, while

$$\begin{aligned} \frac{d}{ds}_{|s=0} \zeta_{\bar{A}}(s) &= (\log c^{2l})\zeta_A(0) + \zeta_A'(0) \\ \Rightarrow e^{-\zeta_{\bar{A}}'(0)} &= e^{-(\log c^{2l})\zeta_A(0) - \zeta_A'(0)} \\ &= c^{-2l\zeta_A(0)} \exp(-\zeta_A'(0)), \text{ that is} \\ \det \bar{A} &= c^{-2l\zeta_A(0)} \det A. \end{aligned}$$

This observation motivates the following definition:

$$P(A_g) := (\mathrm{Vol}(M,g))^{\frac{2l\zeta_A(0)}{n}} \det A. \tag{5.4}$$

Then

$$\begin{aligned} P(\bar{A}_{\bar{g}}) &= (\mathrm{Vol}(M,\bar{g}))^{\frac{2l\zeta_{\bar{A}}(0)}{n}} \det \bar{A} \\ &= (\mathrm{Vol}(M,g))^{\frac{2l\zeta_A(0)}{n}} c^{2l\zeta_A(0)} \det \bar{A} \\ &= (\mathrm{Vol}(M,g))^{\frac{2l\zeta_A(0)}{n}} \det A \\ &= P(A_g), \end{aligned}$$

since $\mathrm{vol}(M,\bar{g}) = c^n \mathrm{Vol}(M,g)$ for $\bar{g} = c^2 g, c > 0$. Thus $P(A_g)$ is a scale invariant quantity.

The following *conformal index theorem* is due to Branson and Orsted [14].

Theorem 5.1 (Branson–Orsted) *Assume that A is as above and conformally covariant (or a positive integral power of conformally covariant operators). For simplicity assume that*

$$N(A) := \#\{j : \lambda_j = 0\} = 0.$$

Then for $a_k(A_g) := a_k(1, A_g)$,

$$\frac{d}{d\varepsilon}_{|\varepsilon=0} a_k(A_{g_{w+\varepsilon f}}) = (n-k) a_k(f, A_{g_w}), \tag{5.5}$$

$$\frac{d}{d\varepsilon}_{|\varepsilon=0} \zeta'_{A_{g_{w+\varepsilon f}}}(0) = 2l a_n(f, A_{g_w}). \tag{5.6}$$

Notice that (5.5) for $k = n$ implies that $a_n(A_{g_w})$ is conformally invariant. We can compute $\zeta'_{A_{g_w}}(0) - \zeta'_{A_g}(0) = -\log \frac{|\det A_{g_w}|}{|\det A_g|} = -\log \frac{\det A_{g_w}}{\det A_g}$, using the fact that the number of negative eigenvalues appearing in the definition (5.3) is conformally invariant for conformally covariant operators.

In terms of the scale invariant quantity $P_A(g)$, the last quotient may be rewritten as

$$-\log \frac{P(A_{g_w})}{P(A_g)} = -\frac{2l\zeta_A(0)}{n} \log \frac{\mathrm{Vol}(M,g_w)}{\mathrm{Vol}(M,g)} - \log \frac{\det A_{g_w}}{\det A_g}.$$

By (5.6) we arrive at

$$\zeta'_{A_{g_w}}(0) - \zeta'_{A_g}(0) = \int_0^1 \frac{d}{dt}\zeta'_{A_{g_{tw}}}(0)\,dt = 2l\int_0^1 a_n(w, A_{g_{tw}})\,dt, \tag{5.7}$$

by the simple identity $\frac{d}{d\varepsilon}|_{\varepsilon=0}\zeta'_{A_{g_{tw+\varepsilon w}}}(0) = \frac{d}{dt}\zeta'_{A_{g_{tw}}}(0)$.

Remark 5.2 When n is odd, $a_n \equiv 0$ for compact closed n-manifolds. Hence $\log\det A_{g_w}$ is a constant, compare to (3.6).

We now focus on the case $n = 4$. Assuming $N(A) = N(A_g) = 0$ as in Theorem 5.1, then we have

Lemma 5.3 *Let A be as in Theorem 5.1 on $(M^4, g_0), M$ compact and closed, with $l = 1$. Then there are constants $\gamma_1, \gamma_2, \gamma_3$ depending on A but not on g_0, such that*

$$B_4(A_g) = \gamma_1|W_g|_g^2 + \gamma_2 Q_g - \gamma_3\Delta_g R_g, \tag{5.8}$$

$$|W_{g_w}|_{g_w}^2 = e^{-4w}|W_{g_0}|_{g_0}^2, \tag{5.9}$$

$$R_{g_w} = e^{-2w}(R_g - 6\Delta_0 w - 6|\nabla_0 w|_{g_0}^2), \tag{5.10}$$

$$\begin{aligned}\Delta_{g_w}R_{g_w} &= \delta_{g_w}d_{g_w}R_{g_w}\\ &= e^{-4w}(\Delta_0 R_{g_0} + b_1(w) + b_2(w) + b_3(w))\end{aligned} \tag{5.11}$$

with

$$\begin{aligned}
b_1(w) &= -6\Delta_0^2 w - 2\Delta_0 w R_{g_0} - 2\langle\nabla_0 w, \nabla_0 R_{g_0}\rangle_{g_0},\\
b_2(w) &= -6\Delta_0(|\nabla_0 w|_{g_0}^2) + 12(\Delta_0 w)^2 + 12\langle\nabla_0 w, \nabla_0\Delta_0 w\rangle_{g_0},\\
b_3(w) &= 12\Delta_0 w|\nabla_0 w|_{g_0}^2 + 12\langle\nabla_0 w, \nabla_0(|\nabla_0 w|_{g_0}^2)\rangle_{g_0},
\end{aligned}$$

where each $b_i(w)$ is homogeneous of degree i in w.

Remarks

1. Recall (4.12), i.e., $12Q_g := R_g^2 - 3|\mathrm{Ric}_g|_g^2 - \Delta_g R_g$. In general, there are only four possible metric invariants of order 4, namely $R_g^2, |\mathrm{Ric}_g|_g^2, |W_g|_g^2$ and $\Delta_g R_g$, a linear combination of which furnishes $B_k(A_g)$. Apart from $|W_g|_g^2$ these are not pointwise conformal invariants, only the integral of them is. Moreover, the conformal covariance of A, i.e., $b - a = 2$, enforces the ratio $R_g^2 : |\mathrm{Ric}_g|_g^2$ to be $1 : -3$, which allows us to express $B_k(A_g)$ in terms of $|W_g|_g^2, \Delta_g R_g$ and Q_g.
2. The negative divergence introduced for the Paneitz operator (Example 3 in Chapter 4) satisfies the covariance relation

$$\delta_{g_w}\alpha = e^{-4w}\delta_g e^{2w\alpha}, \tag{5.12}$$

for any 1-form α, and

$$d_{g_w}f = d_g f \tag{5.13}$$

for any function f.

Sketch of the proof of Lemma 5.3. The fact that (5.8) holds true is made plausible in the first remark above and (5.10) is a direct consequence of the conformal covariance of A. For the conformal Laplacian, $A = L$, one obtains (recall (4.4) in Chapter 4) for $n = 4$,

$$L_{g_w}(\varphi) = e^{-3w} L_{g_0}(e^w \varphi) \quad \text{for all } \varphi \in C^\infty(M),$$

and setting $\varphi \equiv 1$

$$-\Delta_{g_w}(1) + \frac{R_{g_w}}{6} = e^{-3w}(-\Delta_0(e^w) + \frac{R_{g_0}}{6} e^w),$$

which implies (5.10) (for $A = L$), since $\frac{\Delta_0 e^w}{e^w} = \Delta_0 w + |\nabla_0 w|^2_{g_0}$.

The identity (5.11) follows from a straightforward computation using (5.12) and (5.10). □

Recalling (4.11) one deduces from (5.8) – (5.11) that

$$B_4(A_{g_w}) = e^{-4w}(B_4(A_{g_0}) + \frac{1}{2}\gamma_2 (P_4)_{g_0} w - \gamma_3(b_1(w) + b_2(w) + b_3(w)), \tag{5.14}$$

where $(P_4)_{g_0}$ denotes the Paneitz operator with respect to the background metric g_0.

Under the assumption that A does not have zero eigenvalues, i.e., $N(A) = 0$, we can go back to (5.7) to compute the log determinant (for $l = 1$):

$$\begin{aligned}
-\log \frac{\det A_{g_w}}{\det A_{g_0}} &= \zeta'_{A_{g_w}}(0) - \zeta'_{A_{g_0}}(0) = 2 \int_0^1 \left[\int_M w B_4(A_{g_{tw}})\, dv_{g_{tw}} \right] dt \\
&\underset{(5.14)}{=} 2 \int_0^1 \Bigg[\int_M w(B_4(A_{g_0}) + \frac{1}{2}\gamma_2 t (P_4)_{g_0} w \\
&\qquad\qquad - \gamma_3(t b_1(w) + t^2 b_2(w) + t^3 b_3(w)))e^{-4tw}\, dv_{g_{tw}} \Bigg] dt \\
&= 2 \int_M w \left(B_4(A_{g_0}) + \frac{1}{4}\gamma_2 (P_4)_{g_0} w - \gamma_3 \left(\frac{1}{2} b_1(w) + \frac{1}{3} b_2(w) + \frac{1}{4} b_3(w) \right) \right) dv_0,
\end{aligned} \tag{5.15}$$

where we used $dv_{g_{tw}} = e^{4tw}\, dv_0$ and the homogeneity of the $b_i, i = 1,2,3$. In terms of the scale-invariant expression $P(A)$,

$$-\log P(A_{g_w}) + \log P(A_{g_0}) = -\log \frac{\det A_{g_w}}{\det A_{g_0}} - \frac{1}{2}\zeta_A(0) \log \frac{\mathrm{Vol}(M, g_w)}{\mathrm{Vol}(M, g_0)},$$

where

$$\begin{aligned}
\zeta_A(0) = \int_M B_4(A_{g_0})\, dv_0 &\underset{(5.8)}{=} \int_M (\gamma_1 |W_{g_0}|^2_{g_0} + \gamma_2 Q_{g_0} - \gamma_3 \Delta_0 R_{g_0}) dv_0 \\
&= \gamma_1 \int_M |W_{g_0}|^2_{g_0}\, dv_0 + \gamma_2 \int_M Q_{g_0}\, dv_0.
\end{aligned} \tag{5.16}$$

Thus we have

Theorem 5.4 (Branson–Orsted)[14] *Let A be as in Lemma 5.3, then*

$$F_A[w] := -2\log\frac{P(A_{g_w})}{P(A_{g_0})} = \gamma_1 \mathrm{I}[w] + \gamma_2 \mathrm{II}[w] + \gamma_3 \mathrm{III}[w],$$

where

$$\begin{aligned}
\mathrm{I}[w] &:= 4\int_M w|W_{g_0}|^2_{g_0}\,dv_0 - \int_M |W_{g_0}|^2_{g_0}\,dv_0 \log \fint_M e^{4w}\,dv_0,\\
\mathrm{II}[w] &:= \int_M (w(P_4)_{g_0}w + 4wQ_{g_0})\,dv_0 - \int_M Q_{g_0}\,dv_0 \log \fint_M e^{4w}\,dv_0,\\
\mathrm{III}[w] &:= -4\int_M \left(w\Delta_0 R_{g_0} + \frac{1}{2}wb_1(w) + \frac{1}{3}wb_2(w) + \frac{1}{4}wb_3(w)\right) dv_0\\
&= \frac{1}{3}\left(\int_M R^2_{g_w}\,dv_{g_w} - \int_M R^2_{g_0}\,dv_0\right).
\end{aligned}$$

Remarks

1. The last equality in the expression III can be obtained by an integration by parts. Notice that by (5.10), $R^2_{g_w}\,dv_{g_w} = R^2_{g_w}e^{4w}\,dv_0 = (R_{g_0} - 6\Delta_0 w - 6|\nabla_g w|^2_{g_0})^2\,dv_0$.
2. For $A = L = -\Delta + R/6$ the ratios between the γ_i are as follows, see [14],

$$(4\pi)^2 180(\gamma_1,\gamma_2,\gamma_3) = \left(1, -4, -\frac{2}{3}\right).$$

For the square of the Dirac operator $A = \boldsymbol{\nabla}^2$ ($\boldsymbol{\nabla}$ is a conformally covariant operator of bidegree $\left(\frac{5}{2},\frac{3}{2}\right)$) one has

$$(4\pi)^2 360(\gamma_1,\gamma_2,\gamma_3) = \left(-7, 88, \frac{28}{6}\right).$$

Notice that $\gamma_2\gamma_3 > 0$ in both examples.
In Branson's notation [10] our $(\gamma_1,\gamma_2,\gamma_3)$ correspond to $(\beta_1,\beta_2,\beta_3/6)$.

Let us now recall some facts about the Yamabe metric. Given (M^n, g_0) compact without boundary, one defines

$$Y(M^n, g_0) := \inf_{g_w\in[g_0]} \frac{\int_M R_{g_w}\,dv_{g_w}}{\left(\int_M dv_{g_w}\right)^{\frac{n-2}{n}}},$$

which is called the *Yamabe constant*, a conformally invariant quantity. Here $[g_0]$ denotes the class of all metrics that are conformal to the background metric g_0. One central result regarding the Yamabe constant is due to Yamabe [92], Trudinger [88], Aubin [4] and Schoen [82]:

Theorem 5.5

(i) $\mathrm{sign}(Y(M^n, g_0)) = \mathrm{sign}(\lambda_1(L_{g_0}))$, *where* λ_1 *denotes the first eigenvalue of the conformal Laplacian* L_{g_0}.

(ii) $Y(M^n, g_0) \le Y(S^n, g_c)$ *with equality iff* (M^n, g_0) *is conformally equivalent to* (S^n, g_c).

(iii) $Y(M^n, g_0)$ *is attained by some metric* $g_w \in [g_0]$ *with* $R_{g_w} \equiv \mathrm{const}$. *This metric is referred to as the* Yamabe metric *and often denoted by* g_Y.

Proof. Since we are going to need only the first part, we will restrict our attention to proving (i).

Let g_0 be the background metric. For any $u \in C^\infty(M)$, $u > 0$, set $\bar{g}_u := u^{\frac{4}{n-2}} g_0$, then

$$R_{\bar{g}_u} = \frac{1}{C_n} u^{-\frac{n+2}{n-2}} L_{g_0} u,$$

where $L_{g_0} u = -\Delta_0 u + C_n R_{g_0} u$ is the conformal Laplacian, $C_n = \frac{n-2}{4(n-1)}$. It follows that

$$\int_M R_{\bar{g}_u} dv_{\bar{g}_u} = \frac{1}{C_n} \int_M u L_{g_0} u dv_0 = \frac{1}{C_n} \int_M (|\nabla_0 u|^2 + C_n R_{g_0} u^2) dv_0.$$

Let ϕ_1 be the first eigenfunction of L_{g_0} with $||\phi_1||_{L^2(M,g_0)} = 1$. Then $\phi_1 \in C^\infty(M)$ and it does not change sign. We may assume that $\phi_1 > 0$. Note that

$$\frac{\int_M R_{\bar{g}_{\phi_1}} dv_{\bar{g}_{\phi_1}}}{(\int_M dv_{\bar{g}_{\phi_1}})^{\frac{n-2}{n}}} = \frac{\lambda_1}{C_n ||\phi_1||^2_{L^{\frac{2n}{n-2}}(M,g_0)}}.$$

Thus if $\lambda_1 < 0$, then $Y(M^n, g_0) < 0$.

If $\lambda_1 = 0$, then the above formula shows that $Y(M^n, g_0) \le 0$; while we also have $\int_M R_{\bar{g}_u} dv_{\bar{g}_u} = \frac{1}{C_n} \int_M u L_{g_0} u dv_0 \ge 0$ for all $u \ge 0$. Thus $Y(M^n, g_0) \ge 0$. Hence $Y(M^n, g_0) = 0$.

If $\lambda_1 > 0$, then for any $u \in C^\infty(M)$, $u > 0$, $\int_M u L_{g_0} u dv_0 \ge \lambda_1 ||u||^2_{L^2(M,g_0)}$. On the other hand we also have

$$\int_M u L_{g_0} u dv_0 \ge ||u||^2_{H^1(M,g_0)} - C(g_0) ||u||^2_{L^2(M,g_0)}.$$

Thus we have

$$\int_M u L_{g_0} u dv_0 \ge C_{g_0} ||u||^2_{H^1(M,g_0)} \ge C_{g_0} ||u||^2_{L^{\frac{2n}{n-2}}(M,g_0)}$$

by the Sobolev embedding inequality. Hence

$$\frac{\int_M R_{\bar{g}_u} dv_{\bar{g}_u}}{(\int_M dv_{\bar{g}_u})^{\frac{n-2}{n}}} \ge C(g_0) > 0.$$

That is $Y(M^n, g_0) \ge C(g_0) > 0$.

For (ii) and (iii) we refer to [4], [82]. □

Notice that if $Y(M^n, g_0) \geq 0$, then taking the Yamabe metric $g_Y (\Rightarrow R_{g_Y} \equiv$ const. ≥ 0 according to part (iii) of the previous theorem), we are led to the estimate (taking $\mathrm{Vol}(M, g_w) = \mathrm{Vol}(M, g_Y) = 1$ for simplicity),

$$\begin{aligned} \int_M R_{g_w}^2 \, dv_{g_w} &\geq \left(\int_M R_{g_w} \, dv_{g_w} \right)^2 \\ &\geq \left(\int_M R_{g_Y} \, dv_{g_Y} \right)^2 = \int_M R_{g_Y}^2 \, dv_{g_Y} . \end{aligned}$$

Thus III $[w] \geq 0$ for all w in Theorem 5.4, and it is zero only when $R_{g_w} = R_{g_Y}$. We take this as indication that it is very nontrivial to achieve the infimum of III$[w]$.

Before discussing extremal problems for the zeta functional determinant $F[\cdot]$ in Theorem 5.4 on general manifolds, we turn our attention to studying extremal metrics on S^4 with respect to the conformal Laplacian:

Theorem 5.6 (Branson–Chang–Yang) [12] *On* (S^4, g_c) $\det L_{g_w}$ *is minimized for* $g_w = e^{2w} g_c$, *with the volume constraint* $\mathrm{Vol}(S^4, g_w) = \mathrm{Vol}(S^4, g_c) = \frac{8\pi^2}{3} = |S^4|$, *iff* $g_w = \phi^*(g_c)$ *for some conformal transformation* $\phi : S^4 \to S^4$, *i.e.,* g_w *and* g_c *are isometric.*

The theorem above should be viewed as a 4-dimensional analogue of the Onofri inequality in Theorem 3.1 and Corollary 3.2.

Remarks

1. For the Dirac operator $\boldsymbol{\nabla}^2$ one gets $\det \boldsymbol{\nabla}^2_{g_w}$ is maximized iff g_w is isometric to g_c.

2. On (S^4, g_c) one has $|W_{g_c}|^2_{g_c} \equiv 0$, hence $\mathrm{I}[w] \equiv 0$, $\mathrm{II}[w] \geq 0$ with equality iff $g_w = \phi^*(g_c)$ and $\mathrm{III}[w] \geq 0$ as pointed out before, since $g_c = g_Y$, here, with $R_{g_c} \equiv 12$, and equality iff $g_w = \phi^*(g_c)$.

3. We may view the fact that $\mathrm{II}[w] \geq 0$ as a special case of Beckner's inequality [7], stated for general operators P_n on (S^n, g_c), given by

$$(P_n)_{g_c} := \begin{cases} \Pi_{k=0}^{\frac{n-2}{2}} (-\Delta_{g_c} + k(n-k-1)), \text{ for } n \text{ even}, \\ \left(-\Delta_{g_c} + \left(\frac{n-1}{2}\right)^2 \right)^{\frac{1}{2}} \Pi_{k=0}^{\frac{n-3}{2}} (-\Delta_{g_c} + k(n-k-1)), \text{ for } n \text{ odd}. \end{cases}$$

Branson [9] pointed out that these operators P_n may be obtained by conformally pulling back the operator $(-\Delta)^{n/2}$ on $\mathbb{R}^n$ via stereographic projection $\pi : S^n - \{\mathbb{N}\} \to \mathbb{R}^n$; where $\mathbb{N}$ denotes the north pole of the sphere S^n. For instance, for $n = 2$ one obtains the Laplacian Δ_{g_c} on S^2 by conformally pulling back $-\Delta$ on $\mathbb{R}^2$, whereas for $n = 4$ one gets the Paneitz operator

$$(P_4)_{g_c} = (-\Delta_{g_c}) \left(-\Delta_{g_c} + \frac{1}{6} R_{g_c} \right) = (-\Delta_{g_c})(-\Delta_{g_c} + 2);$$

compare with Example 3b of Chapter 4.

Beckner's inequality states

$$\log \fint_{S^n} e^{nw}\, dv_{g_c} \leq n \fint_{S^n} w\, dv_{g_c} + \frac{n}{2(n-1)!} \fint_{S^n} w P_n(w)\, dv_{g_c}$$

with equality iff $g_w = \phi^*(g_c)$.

For $n = 2$ this reduces to Onofri's inequality (Theorem 2.11), while for $n = 4$ Beckner's inequality implies $\mathrm{II}[w] \geq 0$, since on $(S^4, g_c), Q_{g_c} \equiv 3$ according to (4.12) with $R_{g_c} \equiv 12$.

4. For more general results we give the foolowing overview:

standard metric g_c on	is a	for the operator	among metrics with fixed	proved by
S^2	global max global min	$\det(-\Delta)$ $\det \boldsymbol{\nabla}^2$	area area	Onofri [72]
S^4	global min global max	$\det L$ $\det \boldsymbol{\nabla}^2$	volume & conformal class	Branson, Chang, Yang [12]
S^6	global max global min	$\det L$ $\det \boldsymbol{\nabla}^2$	volume & conformal class	Branson [11]
S^3	local max local max	$\det(-\Delta)$ $\det(-\Delta)$	volume & conformal class volume	K. Richardson [80] K. Okikiolu [70]
$S^{2n+1}, n \geq 3$	saddle point	$\det(-\Delta)$	volume & conformal class	K. Okikiolu [70]
S^{4n+1} S^{4n+3}	local min local max	$\det L$ $\det L$	volume	K. Okikiolu [70]

Here L denotes the conformal Laplace operator. The results by Okikiolu, [70] especially the result that on the 3-sphere S^3. $\det(-\Delta_{g_c})$ is a local maximum of the functional $\det(-\Delta_g)$ among *all* metrics g (not only the ones conformal to g_c) defined on S^3, are truly remarkable. An important tool in her work is the computation of the *canonical trace* of odd operators in odd dimensions. In a separate paper [69], she has also given an alternative proof of Polyakov's formula, Theorem 3.1, using the calculus of pseudo-differential operators.

§ 6 Extremal metrics for the log-determinant functional

We study the extremal metric for the functional $F_A[w]$ given in Theorem 5.4 by Branson and Orsted. As a basic tool we will need the following generalization of Moser's inequality, *Adams' inequality.*

Lemma 6.1 (Adams [1]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and suppose $k < n$. Then there are constants $c = c(k,n), \beta_0 = \beta_0(k,n)$, such that for all $w \in C_0^k(\Omega)$ with $||\nabla^k w||_p \le 1, p = \frac{n}{k}$, we have*

$$\int_\Omega \exp(\beta |w(x)|^{p'})\, dx \le c|\Omega| \tag{6.1}$$

for all $\beta \le \beta_0$, and $p' \colon = p/(p-1)$.

This inequality is sharp in the following sense: If $\beta > \beta_0$, then for any $N \in \mathbb{N}$ there exists $u_N \in C_0^\infty(\Omega)$ with $||\nabla^k u_N||_p \le 1$, such that

$$\int_\Omega \exp(\beta |u_N(x)|^{p'})\, dx > N|\Omega|.$$

Notice that we denote

$$\begin{aligned} ||\nabla^k u|| \colon &= ||\Delta^{k/2} u|| \qquad \text{for } k \text{ even},\\ ||\nabla^k u|| \colon &= ||\nabla \Delta^{\frac{k-1}{2}} u|| \ \text{ for } k \text{ odd}. \end{aligned}$$

If $n = 4, k = 2$, whence $p = p' = 2$, then $\beta_0 = \beta_0(2,4) = 32\pi^2$. On a compact 4-manifold, Lemma 6.1 takes the following form (cf. [12], [46] for general M^n):

Lemma 6.2 *On (M^4, g_0) compact, closed, there exists a constant $c_0 = c_0(g_0)$ such that for all $w \in C^2(M)$ with $||\Delta_0 w||_2 \le 1$,*

$$\int_M \exp(32\pi^2 |w - \bar{w}|^2)\, dv_0 \le c_0. \tag{6.2}$$

Corollary 6.3 *On (M^4, g_0) as above one has*

$$\log \fint_M e^{4(w-\bar{w})}\, dv_0 \le \log c_0 + \frac{1}{8\pi^2} ||\Delta_0 w||_2^2. \tag{6.3}$$

(6.3) follows from (6.2) in the same way as Corollary 1.7 was deduced from Corollary 1.6 in the first chapter.

Define for a metric g on M,

$$k_g \colon = \int_M Q_g\, dv_g, \tag{6.4}$$

which is a conformally invariant constant, i.e., $k_g = k_{g_0} = \int_M Q_0\, dv_0$ for $g = g_w = e^{2w} g_0$. Due to the Chern–Gauss–Bonnet formula

$$4\pi^2 \chi(M^4) = \frac{1}{8} \int_M |W|^2\, dv + \int_M Q\, dv. \tag{6.5}$$

Suppose in the following that $\gamma_2 < 0$ in the representation of $F_A[w]$ given in Theorem 5.4 (otherwise consider $(-F_A)$ instead).

Lemma 6.4 *Assume that $\gamma_2 < 0$, and $\gamma_2\gamma_3 > 0$. Let $c_1, c_2 \in \mathbb{R}$ be given constants with $c_2 > 0$ and suppose that*

$$k_{g_0} < 8\pi^2 - \frac{\gamma_1}{\gamma_2} \int_M |W_0|_0^2\, dv_0. \tag{6.6}$$

Then for all $w \in \mathcal{S}_{c_1,c_2}(A)$, where

$$\mathcal{S}_{c_1,c_2}(A) \colon = \{w \in C^\infty(M) \colon (\operatorname{sign} \gamma_2) F_A[w] \le c_1, \operatorname{vol}(M, g_w) = c_2 \operatorname{vol}(M, g_0)\},$$

one has the uniform estimate

$$||w||_{W^{2,2}} \le C(c_1, c_2, A, g_0). \tag{6.7}$$

Remark. If we assume for simplicity that $A = L$, as we did in the proof of Lemma 5.3, we have

$$(4\pi)^2 180(\gamma_1, \gamma_2, \gamma_3) = \left(1, -4, -\frac{2}{3}\right),$$

according to the second remark following Theorem 5.4. Hence the condition on k_{g_0} in Lemma 6.4 reads as

$$k_{g_0} < 8\pi^2 + \frac{1}{4} \int_M |W_0|_0^2\, dv_0.$$

Proof of Lemma 6.4. We will show that, under the assumptions $\gamma_2\gamma_3 > 0$ and (6.6), the terms II $[w]$ and III $[w]$ in the representation for $F_A[w]$ add up to some multiple of the $W^{2,2}$-norm of w. All the terms involving the background metric g_0 will carry a sub- or superscript "0", whereas $g = g_w = e^{2w} g_0$ will not be indicated explicitly, e.g., $\nabla_{g_0} = \nabla_0$, but $\nabla_g = \nabla$.

$$\begin{aligned}
\mathrm{II}[w] &= \int_M (w, P_{4\,0} w)_0\, dv_0 + 4 \int_M Q_0 (w - \bar{w})\, dv_0 \\
&\quad - \int_M Q_0\, dv_0 \log \fint_M e^{4(w-\bar{w})}\, dv_0 \\
&\underset{(4.9)}{=} \int_M (\Delta_0 w)^2\, dv_0 + \frac{2}{3} \int_M R_0 |\nabla_0 w|_0^2\, dv_0 \\
&\quad - 2 \int_M \mathrm{Ric}_0(\nabla_0 w, \nabla_0 w)\, dv_0 + 4 \int Q_0 (w - \bar{w}) dv_0 \\
&\quad - \int_M Q_0\, dv_0 \log \fint_M e^{4(w-\bar{w})}\, dv_0.
\end{aligned} \tag{6.8}$$

For III$[w]$ one computes

$$
\begin{aligned}
\mathrm{III}[w] &= \frac{1}{3}\left(\int_M R^2\,dv - \int_M R_0^2\,dv_0\right) \\
&= \frac{1}{3}\int_M [36(\Delta_0 w + |\nabla_0 w|_0^2)^2 - 12R_0(\Delta_0 w + |\nabla_0 w|_0^2)]\,dv_0 \qquad (6.9) \\
&= 12\int_M (\Delta_0 w + |\nabla_0 w|_0^2)^2\,dv_0 - 4\int_M R_0(\Delta_0 w + |\nabla_0 w|_0^2)\,dv_0,
\end{aligned}
$$

where we used (5.10); compare with Remark 1 after Theorem 5.4. The assumption on k_{g_0} may be rewritten as

$$
-\gamma_2\int_M Q_0\,dv_0 - \gamma_1\int_M |W_0|_0^2\,dv_0 < -\gamma_2 8\pi^2, \qquad (6.10)
$$

since $\gamma_2 < 0$. This implies by (6.3)

$$
\begin{aligned}
\Big[-\gamma_2\int_M Q_0\,dv_0 - \gamma_1\int_M |W_0|_0^2\,dv_0\Big]\log ⨍_M e^{4(w-\bar{w})}\,dv_0& \\
< -\gamma_2 8\pi^2\left(\frac{1}{8\pi^2}\int_M (\Delta_0 w)^2\,dv_0 + c_0\right)& \qquad (6.11) \\
= -\gamma_2\int_M (\Delta_0 w)^2\,dv_0 - 8\pi^2\gamma_2 c_0.&
\end{aligned}
$$

Because of the strict inequality in (6.10) we may rewrite the left-hand side of (6.11) as

$$
\begin{aligned}
\Big[-\gamma_2\int_M Q_0\,dv_0 - \gamma_1\int_M |W_0|_0^2\,dv_0\Big]\log ⨍_M e^{4(w-\bar{w})}\,dv_0& \\
\le (-\gamma_2 - \varepsilon)\int_M (\Delta_0 w)^2\,dv_0 + C& \qquad (6.12)
\end{aligned}
$$

for some $\varepsilon > 0$.

Inserting (6.8), (6.9) and (6.12) into the expression for $F_A[w]$ we can estimate

$$
\begin{aligned}
F_A[w] \le{}& (\gamma_2 + 12\gamma_3 - \gamma_2 - \varepsilon)\int_M (\Delta_0 w)^2\,dv_0 \\
&+ 24\gamma_3\int_M (\Delta_0 w)|\nabla_0 w|_0^2\,dv_0 + 12\gamma_3\int_M |\nabla_0 w|^4\,dv_0 \\
&+ \text{ lower order terms in } w.
\end{aligned}
$$

Since $\varepsilon > 0, \gamma_2 < 0, \gamma_2\gamma_3 > 0$, we obtain by Young's inequality and the Sobolev embedding $W^{1,4} \hookrightarrow W^{2,2}$, that first

$$
\int_M |\nabla_0 w|_0^4\,dv_0 \le C(c_1, c_2, F_A[w]),
$$

and then

$$
\int_M (\Delta_0 w)^2\,dv_0 \le C(c_1, c_2, F_A[w]). \qquad \square
$$

Lemma 6.4 now implies

Theorem 6.5 ([33]) *If $\gamma_2 < 0, \gamma_2\gamma_3 > 0$, and if*

$$k_{g_0} < 8\pi^2 - \frac{\gamma_1}{\gamma_2} \int_M |W_0|_0^2 \, dv_0,$$

then there exists an extremal metric $g = g_w = e^{2w} g_0$ with $w \in W^{2,2}(M)$,

$$F_A[w] = \sup_{\mathcal{S}_{c_1,c_2}(A)} F_A[\cdot],$$

satisfying (in terms of the metric g)

$$\gamma_1 |W|^2 + \gamma_2 Q - \gamma_3 \Delta R = \gamma_1 \int_M |W|^2 \, dv + \gamma_2 \int_M Q \, dv \equiv \text{const.} \tag{6.13}$$

Furthermore, $w \in C^\infty(M)$ according to [25].

Notice that this result applies to the conformal Laplacian $A \colon = L$, where $(\gamma_1, \gamma_2, \gamma_3) \sim (1, -4, -2/3)$, if $k_{g_0} < 8\pi^2 + (1/4) \int_M |W_0|_0^2 \, dv_0$.

Regarding regularity even more is true:

Theorem 6.6 (Uhlenbeck–Viaclovsky [89]) *Any critical point of $F_A[\cdot]$ of class $W^{2,2}(M)$ is C^∞-smooth.*

Our next goal is to derive an application of Theorem 6.5 given by Gursky, see Theorem 6.7. Denote

$$\sigma_2 \colon = \frac{1}{2} \left(\frac{1}{12} R^2 - |E|^2 \right) \tag{6.14}$$

(in terms of some metric g on M), where E is the Einstein tensor on M, and recall the identity

$$\mathrm{Ric} = E + \frac{R}{4} g, \tag{6.15}$$

to conclude by (4.12), and the fact that $TrE \equiv 0$,

$$\begin{aligned} 12Q &= -\Delta R + R^2 - 3|\mathrm{Ric}|^2 \\ &\underset{(6.15)}{=} -\Delta R + \frac{1}{4} R^2 - 3|E|^2 \\ &= -\Delta R + 3 \left(\frac{1}{12} R^2 - |E|^2 \right) \\ &= -\Delta R + 6\sigma_2. \end{aligned} \tag{6.16}$$

(The notation σ_2 is motivated by more general considerations regarding elementary symmetric functions σ_k of the eigenvalues of geometric tensors, see Chapter 7.)

Two alternative formulations of Theorem 6.5 turn out quite useful later on:

Theorem 6.5′ If $\gamma_2, \gamma_3 < 0$, and if

$$k_{g_0} = \int_M Q_0 \, dv_0 < 8\pi^2 - \frac{\gamma_1}{\gamma_2} \int_M |W_0|_0^2 \, dv_0,$$

or equivalently, if

$$k_d \colon = \gamma_1 \int_M |W_0|_0^2 \, dv_0 + \gamma_2 \int_M Q_0 \, dv_0 > \gamma_2 8\pi^2,$$

then there is $w_d \in C^\infty(M)$ such that

$$F_A[w_d] = \sup_{\mathcal{S}_{c_1,c_2}(A)} F_A[\cdot],$$

and in terms of the metric $g = g_{w_d} = e^{2w_d} g_0$,

$$\gamma_1 |W|^2 + \gamma_2 Q - \gamma_3 \Delta R \equiv \frac{k_d}{\mathrm{vol}(M, g_{w_d})}. \tag{6.17}$$

As it is sometimes more convenient to take γ_2 and γ_3 to be positive numbers instead of negative numbers; we may take $\inf F_A$ instead of $\sup F_A$ and restate Theorem 6.5′ as:

Theorem 6.5″ If $\gamma_2, \gamma_3 > 0, k_d < \gamma_2 8\pi^2$, then there exists $w_d \in C^\infty(M)$ with

$$F_A[w_d] = \inf_{\mathcal{S}_{c_1,c_2}(A)} F_A[\cdot],$$

such that in terms of the metric $g = g_{w_d} = e^{2w_d} g_0$, (6.17) holds, or equivalently,

$$\begin{aligned} \gamma_1 |W|^2 + \gamma_2 \left(-\frac{1}{12} \Delta R + \frac{1}{2} \sigma_2 \right) - \gamma_3 \Delta R &= \frac{k_d}{\mathrm{vol}(M, g_{w_d})} \\ \Leftrightarrow - \left(\frac{1}{12} \gamma_2 + \gamma_3 \right) \Delta R &= -\gamma_1 |W|^2 - \frac{1}{2} \gamma_2 \sigma_2 + \frac{k_d}{\mathrm{vol}(M, g_{w_d})} \\ \Leftrightarrow \Delta R &= \lambda + \alpha |W|^2 + \beta \sigma_2, \end{aligned} \tag{6.18}$$

where

$$\lambda \colon = -\frac{k_d}{\mathrm{vol}(M, g_{w_d})} \left(\frac{1}{12} \gamma_2 + \gamma_3 \right)^{-1} \leq 0,$$

$$\alpha \colon = \gamma_1 \left(\frac{1}{12} \gamma_2 + \gamma_3 \right)^{-1} \leq 0, \text{ and where}$$

$$\beta \colon = \frac{1}{2} \gamma_2 \left(\frac{1}{12} \gamma_2 + \gamma_3 \right)^{-1}.$$

Theorem 6.7 (Gursky [54]) *If* $Y(M^4, g_0) > 0$*, and if* $k_{g_0} \geq 0$*, then the Paneitz operator* $(P_4)_{g_0} = P_0$ *is positive, with* $\lambda_1(P_0) = 0$ *and* $\ker(P_0) = \{\mathbb{R}\}$.

Remarks

1. Both $Y(M^4, g)$ and k_g are conformally invariant quantities, hence the assumptions above are natural, since P_4 is conformally covariant of bidegree $(0, 4)$, see (4.10). This implies that $\ker(P)$ is a conformally invariant set.
2. The proof of Theorem 6.7 will be used to prove the main result in Chapter 7.
3. It is unclear, whether the assumptions $Y(M^4, g_0) > 0, k_{g_0} \geq 0$ are also necessary to obtain P_0 to be positive. Notice that there are indeed Paneitz operators with some negative eigenvalues. For instance, let Σ be the genus 2 hyperbolic surface and $M\colon = \Sigma \times \Sigma$ with $\lambda_1(\Delta_\Sigma) \ll 1$ and $-6 \equiv R < 0$. Then $P = (-\Delta)(-\Delta + (R/6)) = \Delta^2 + \Delta$, which gives
$$\lambda_1(P) = \lambda_1^2(\Delta_\Sigma) - \lambda_1(\Delta_\Sigma) < 0.$$

Before proving Theorem 6.7 we need to derive a few auxiliary results.

Lemma 6.8 *Suppose that* $Y(M^4, g_0) > 0$*, and assume that* (6.18) *holds with* $\alpha \leq 0, 0 \leq \beta \leq 4, \lambda \leq 0$*; then* $R\colon = R_{g_{w_d}} > 0$.

Proof. We are going to show that under these assumptions we actually obtain (in terms of $g = g_{w_d} = e^{2w_d} g_0$)
$$LR \geq 0, \tag{6.19}$$
where $L = L_{g_{w_d}}$ is the conformal Laplacian on (M^4, g_{w_d}) as discussed in Example 2 of Chapter 4. To see that (6.19) holds, recall that for $\psi \in C^2(M)$,
$$L\psi = -\Delta\psi + \frac{R}{6}\psi,$$
so if $\beta \in [0, 4]$, then
$$\begin{aligned} LR &= -\Delta R + \frac{R^2}{6} \\ &\underset{(6.18)}{=} -\lambda - \alpha|W|^2 - \beta\left(\frac{1}{2}\left(\frac{1}{12}R^2 - |E|^2\right)\right) + \frac{R^2}{6} \\ &\geq 0. \end{aligned}$$

Now Lemma 6.8 follows from (6.19) and the following general result. □

Lemma 6.9 *If on* (M^n, g)
$$LR = -\Delta R + c_n R^2 \geq 0 \tag{6.20}$$
(*all in terms of the metric* g), $c_n = \frac{n-2}{4(n-1)}$, *then* $Y(M^n, g) > 0$ *implies* $R = R_g > 0$ *on* M^n.

Proof. Let μ_1 be the first eigenvalue of L and φ the first eigenfunction, $\varphi > 0$. Then we know from Theorem 5.5 (i), that $Y(M^n, g) > 0 \Leftrightarrow \mu_1 > 0$. Defining $f\colon = R/\varphi$ we compute (in terms of g)

$$\begin{aligned} c_n R^2 \underset{(6.20)}{\geq} \Delta R &= \Delta(f\varphi) \\ &= f\Delta\varphi + \varphi\Delta f + 2\langle \nabla f, \nabla\varphi\rangle \\ &= f(c_n R - \mu_1)\varphi + \varphi\Delta f + 2\langle \nabla f, \nabla\varphi\rangle \\ &= c_n R^2 - R\mu_1 + \varphi\Delta f + 2\langle \nabla f, \nabla\varphi\rangle, \end{aligned}$$

i.e., $R\mu_1 \geq \varphi\Delta f + 2\langle \nabla f, \nabla\varphi\rangle$, or

$$f\mu_1 - \frac{2}{\varphi}\langle \nabla f, \nabla\varphi\rangle \geq \Delta f.$$

Since $\mu_1 > 0$, we can apply the minimum principle for f to obtain $f \geq 0$, hence $R \geq 0$. If $f = 0$ at some point, we would get $f \equiv 0$, i.e., $R \equiv 0$ by the strong maximum principle, contradicting $Y(M^n, g) > 0$ (see Theorem 5.5), whence $R > 0$. □

Lemma 6.10 *Let (M^4, g) be a smooth, compact closed 4-manifold. Then $Y(M^4, g) \geq 0$ implies $k_g \leq 8\pi^2$ with equality iff (M^4, g) is conformally equivalent to (S^4, g_c).*

Remarks

1. If $\gamma_2 < 0$ and $Y(M^4, g) \geq 0, \gamma_1 > 0$, then it follows from Lemma 6.10 that the assumptions of Theorem 6.5′ are automatically satisfied unless (M^4, g) is conformally equivalent to (S^4, g_c), in which case the existence result is known anyway.
2. Gursky gave a proof of Lemma 6.10 in [54] without using the fact that $Y(M^4, g) \leq Y(S^4, g_c)$, which we have used in our proof below.

Proof of Lemma 6.10. Using (6.16) we may write (in terms of g)

$$k_g = \int_M Q\, dv = \int_M \frac{1}{4}\left(\frac{1}{12}R^2 - |E|^2\right) dv \leq \frac{1}{48}\int_M R^2\, dv.$$

Since k_g is conformally invariant we may assume that $g = g_Y$, the Yamabe metric, for which $R \equiv R_{g_Y} \equiv$ const. according to Theorem 5.5 (iii). Consequently,

$$\begin{aligned} \int_M R^2\, dv &= R^2 \operatorname{vol}(M, g) \\ &= \left(\int_M R\, dv\right)^2 / \operatorname{vol}(M, g) \\ &= Y(M^4, g)^2 \leq Y(S^4, g_c)^2. \end{aligned}$$

Thus we obtain

$$k_g \leq \frac{1}{48} Y(S^4, g_c)^2 = 8\pi^2$$

with equality iff $Y(M^4, g) = Y(S^4, g_c)$, i.e., iff (M^4, g) is conformally equivalent to (S^4, g_c) by Theorem 5.5 (ii). □

Lemma 6.11 *Let $Y(M^4, g_0) > 0$ and $k_{g_0} \geq 0$. Then there exists $w \in C^\infty(\Omega)$, such that in terms of $g\colon = g_w = e^{2w} g_0$,*

$$\Delta R = \lambda + 2\sigma_2 \tag{6.21}$$

for some $\lambda \leq 0$, where $R = R_{g_w} > 0$.

Proof. Taking $\gamma_1 = 0, \gamma_2 = 6, \gamma_3 = 1$ in Theorem 6 we obtain $w \in C^\infty(M)$ with

$$\Delta R = \lambda + 2\sigma_2.$$

Notice that our assumption $Y(M^4, g_0) > 0$ implies $k_{g_0} \leq 8\pi^2$, and we may assume $k_{g_0} < 8\pi^2$, since otherwise (M^4, g_0) is conformally equivalent to (S^4, g_c), on which (6.21) holds trivially with $|E_{g_c}|^2_{g_c} \equiv 0, R^2_{g_c} \equiv 144 = -12\lambda \Leftrightarrow \lambda = -12$. Note also that the assumption $k_{g_0} \geq 0$ implies $\lambda \leq 0$ by definition of λ in (6.18). Since $\beta = 2$ here, we can apply Lemma 6.8 to obtain $R > 0$. □

Proof of Theorem 6.7. By Lemma 6.11 there is a metric $g = e^{2w} g_0$, such that (in terms of g)

$$\begin{aligned} \Delta R &= \lambda + 2\sigma_2 \\ &= \lambda - |E|^2 + \frac{1}{12} R^2 \end{aligned} \tag{6.22}$$

with $\lambda \leq 0$ and $R > 0$. We can write (again in terms of g), for $\varphi \in C^2(M)$,

$$\begin{aligned} \langle P\varphi, \varphi \rangle_{L^2(dv)} &= \int_M (\Delta\varphi)^2 \, dv + \frac{2}{3} \int_M R|\nabla\varphi|^2 \, dv - 2 \int_M \operatorname{Ric}(\nabla\varphi, \nabla\varphi) \, dv \\ &= \int_M (\Delta\varphi)^2 \, dv + \frac{1}{6} \int_M R|\nabla\varphi|^2 \, dv - 2 \int_M E(\nabla\varphi, \nabla\varphi) \, dv. \end{aligned}$$

Claim

$$2 \int_M E(\nabla\varphi, \nabla\varphi) \, dv \leq \int_M (\Delta\varphi)^2 \, dv + \frac{1}{48} \int_M R|\nabla\varphi|^2 \, dv. \tag{6.23}$$

Before proving the claim notice that then

$$\langle P\varphi, \varphi \rangle_{L^2(dv)} \geq \frac{7}{48} \int_M R|\nabla\varphi|^2 \, dv,$$

which proves Theorem 6.7. □

It remains to show (6.23). The following general fact (see [85], p. 234) is useful:

Lemma 6.12 *Let $M = (m_{ij})$ be an $(n \times n)$-matrix with vanishing trace and norm*

$$|M|^2 \colon = \left(\sum_{i,j=1}^{n} m_{ij}^2 \right)^{\frac{1}{2}} .$$

Then

$$\max_{v \in S^{n-1}} |Mv|^2 \le \frac{n-1}{n} |M|^2. \tag{6.24}$$

To prove (6.23) we take $n = 4$, i.e.,

$$\begin{aligned}
2 \int_M E(\nabla\varphi, \nabla\varphi)\, dv &\underset{(6.24)}{\le} 2\frac{\sqrt{3}}{2} \int_M |E||\nabla\varphi|^2\, dv \\
&\le 2 \int_M \frac{|E|^2}{R} |\nabla\varphi|^2\, dv + \frac{3}{8} \int_M R|\nabla\varphi|^2\, dv \\
&\underset{(6.22)}{=} 2 \int_M \frac{|\nabla\varphi|^2}{R}(-\Delta R + \lambda)\, dv + \frac{13}{24} \int_M R|\nabla\varphi|^2\, dv \\
&\le -2 \int_M |\nabla\varphi|^2 \left(\frac{\Delta R}{R} \right) dv + \frac{13}{24} \int_M R|\nabla\varphi|^2\, dv,
\end{aligned} \tag{6.25}$$

where we used $\lambda \le 0, R > 0$. To estimate the first term we integrate by parts:

$$\begin{aligned}
\int_M |\nabla\varphi|^2 \left(\frac{\Delta R}{R} \right) dv &= - \int_M |\nabla\varphi|^2 \nabla \left(\frac{1}{R} \right) \nabla R\, dv - \int_M \nabla(|\nabla\varphi|^2) \frac{\nabla R}{R}\, dv \\
&\ge \int_M \frac{|\nabla\varphi|^2 |\nabla R|^2}{R^2}\, dv - 2 \int_M \frac{|\nabla R|}{R} |\nabla\varphi||\nabla^2\varphi|\, dv \\
&\ge - \int_M |\nabla^2\varphi|^2\, dv.
\end{aligned}$$

Inserting this into (6.25) we arrive at

$$2 \int_M E(\nabla\varphi, \nabla\varphi)\, dv \le 2 \int_M |\nabla^2\varphi|^2\, dv + \frac{13}{24} \int_M R|\nabla\varphi|^2\, dv. \tag{6.26}$$

Now apply Bochner's formula to get

$$\begin{aligned}
\int_M |\nabla^2\varphi|^2\, dv &= \int_M (\Delta\varphi)^2\, dv - \int_M \mathrm{Ric}(\nabla\varphi, \nabla\varphi)\, dv \\
&= \int_M (\Delta\varphi)^2\, dv - \int_M E(\nabla\varphi, \nabla\varphi)\, dv - \frac{1}{4} \int_M R|\nabla\varphi|^2\, dv.
\end{aligned} \tag{6.27}$$

Substituting (6.27) into (6.26) leads to

$$2\int_M E(\nabla\varphi,\nabla\varphi)\,dv \le 2\int_M (\Delta\varphi)^2\,dv - 2\int_M E(\nabla\varphi,\nabla\varphi)\,dv + \frac{1}{24}\int_M R|\nabla\varphi|^2\,dv,$$

which implies

$$2\int_M E(\nabla\varphi,\nabla\varphi)\,dv \le \int_M (\Delta\varphi)^2\,dv + \frac{1}{48}\int_M R|\nabla\varphi|^2\,dv. \qquad \square$$

For our investigations in Chapters 7 and 8 recall the functional

$$F_A[w] = \gamma_1 \mathrm{I}[w] + \gamma_2 \mathrm{II}[w] + \gamma_3 \mathrm{III}[w]$$

as given in Theorem 5.4. The critical points of $F_A[\cdot]$ satisfy (6.17), i.e., in terms of the corresponding metric $g\colon = g_{w_d} = e^{2w_d} g_0$,

$$-\left(\frac{1}{12}\gamma_2 + \gamma_3\right)\Delta R = -\gamma_1 |W|^2 - \frac{1}{2}\gamma_2\sigma_2 + \frac{k_d}{\mathrm{vol}(M,g)},$$

where $k_d\colon = \gamma_1 \int_M |W_0|_0^2\,dv_0 + \gamma_2 \int_M Q_0\,dv_0$.

If one chooses $\gamma_2 = 1, \gamma_3 = \frac{1}{24}(3\delta - 2), \delta > 0$, and finally γ_1, such that $k_d = 0$, then the Euler–Lagrange equations for the functional

$$F^\delta[w]\colon = \gamma_1 \mathrm{I}[w] + \mathrm{II}[w] + \frac{1}{24}(3\delta - 2)\mathrm{III}[w]$$

read as (in terms of g)

$$\delta\Delta R = 8\gamma_1 |W|^2 + 4\sigma_2, \qquad (*)_\delta$$

or equivalently, (for $\sigma_2 = \sigma_2(A_g)$ as in Chapter 7)

$$\sigma_2(A_g) = \frac{\delta}{4}\Delta R - 2\gamma_1 |W|^2. \qquad (*)'_\delta$$

Notice that if $\int_M \sigma_2(A_g)\,dv \ge 0$, then $\gamma_1 \le 0$ (since $k_d = 0$), and $\gamma_2 = 1$, $\gamma_3 > 0$, if $\delta > \frac{2}{3}$, thus $\gamma_2\gamma_3 > 0$; while $\gamma_1 \le 0$ implies that $\alpha \le 0$ in (6.18), thus we may apply Theorem 6.7, or more precisely Lemma 6.8 to the solution of the equation $(*)_\delta$. Also the equations $(*)_\delta$, $(*)'_\delta$ may be viewed as a δ-regularization of the equation

$$\sigma_2(A_g) = -2\gamma_1 |W|^2 \ge 0$$

for $\gamma_1 \le 0$. That is, a regularization (depending on the parameter δ) of an equation prescribing $\sigma_2(A_g)$. The strategy later will be to let δ tend to zero.

Using the expressions for $\mathrm{I}[w], \mathrm{II}[w], \mathrm{III}[w]$, given in Theorem 5.4 together with (5.10) and (4.9) one can expand $F^\delta[w]$ in terms of derivatives of w with respect to the background metric g_0:

$$
\begin{aligned}
F^\delta[w] = F_0^\delta[w]\colon = & \int_M \left(3\delta(\Delta_0 w)^2 + 3(3\delta-2)\Delta_0 w|\nabla_0 w|^2\right) dv_0 \\
& + \int_M 2(3\delta-2)|\nabla_0 w|^4\, dv_0 \\
& + \text{ lower order terms.}
\end{aligned}
\tag{6.28}
$$

Lemma 6.13 *Let $\mathcal{L}^\delta$ denote the linearization of $(*)_\delta$, i.e., the bilinearization of $F^\delta[\cdot]$ at a critical $w \in W^{2,2}(M)$ with metric $g = g_w = e^{2w}g_0, R_g > 0$. Then, in terms of $g, (dv = dv_g)$,*

$$
\begin{aligned}
\langle \varphi, \mathcal{L}^\delta \varphi\rangle_{L^2(dv)}\colon = & \frac{d^2}{dt^2}_{|t=0} F^\delta[w+t\varphi] \\
= \int_M \left(3\delta(\Delta\varphi)^2 - 4E(\nabla\varphi,\nabla\varphi) + (1-\delta)R|\nabla\varphi|^2\right) dv.
\end{aligned}
\tag{6.29}
$$

Proof. To simplify the computation, notice that the functional $F^\delta[\cdot]$ can be written as

$$F^\delta[w+t\varphi] = F^\delta[w] + F_w^\delta[t\varphi],$$

where $F_w^\delta[\cdot]$ is given by (6.28) with the background metric g_0 replaced by $g = g_w = e^{2w}g_0$. This implies that

$$\frac{d^2}{dt^2}_{|t=0} F^\delta[w+t\varphi] = \frac{d^2}{dt^2}_{|t=0} F_w^\delta[t\varphi].$$

Without loss of generality we may normalize the volume

$$⨍_M e^{4w}\, dv_0 = ⨍_M dv = 1,$$

to obtain by a straightforward computation (in terms of g)

$$
\begin{aligned}
\frac{d^2}{dt^2}_{|t=0} F_w^\delta[t\varphi] = & 16k_d \left(⨍_M \varphi^2\, dv - \left(⨍_M \varphi\, dv\right)^2 \right) + 2\gamma_2 \langle P\varphi, \varphi\rangle_{L^2(dv)} \\
& + 24\gamma_3 \left(\int_M (\Delta\varphi)^2\, dv - \frac{1}{3}\int_M R|\nabla\varphi|^2\, dv \right).
\end{aligned}
$$

Under our hypotheses that $k_d = 0$ (by choice of $\gamma_1 \le 0$), $\gamma_2 = 1, \gamma_3 = \frac{1}{24}(3\delta - 2)$, we get

$$
\begin{aligned}
\frac{d^2}{dt^2}_{|t=0} F_w^\delta[t\varphi] = & 2(\gamma_2 + 12\gamma_3)\int_M (\Delta\varphi)^2\, dv + \frac{4}{3}(\gamma_2 - 6\gamma_3)\int_M R|\nabla\varphi|^2\, dv \\
& - 4\gamma_2 \int_M \mathrm{Ric}(\nabla\varphi,\nabla\varphi)\, dv \\
= & \int_M \left(3\delta(\Delta\varphi)^2 - 4E(\nabla\varphi,\nabla\varphi) + (1-\delta)R|\nabla\varphi|^2\right) dv. \qquad \square
\end{aligned}
$$

We conclude this section with an estimate for the operator $\mathcal{L}^\delta$.

Proposition 6.14 *Let $\mathcal{L}^\delta$ be as in the previous lemma; then, at a solution w with $R = R_{g_w} > 0$, one has for all $\varphi \in W^{2,2}(M)$,*

$$\langle \varphi, \mathcal{L}^\delta \varphi \rangle_{L^2(dv)} \geq \frac{3}{4} \int_M (\delta^2 (\Delta\varphi)^2 + \frac{\delta}{16} R |\nabla \varphi|^2)\, dv.$$

In particular, $\mathcal{L}^\delta \geq 0$ and $\ker \mathcal{L}^\delta = \mathbb{R}$ for all $\delta \geq 0$.

The proof is similar to the one of Theorem 6.7, in particular like the proof of (6.23), recovering Gursky's result "$P \geq 0$" for $\delta = 2/3$.

In Chapter 8 we will use a continuity method to let $\delta \to 0$ in $(*)_\delta$. Proposition 6.14 will serve us to prove the openness for the continuity argument.

§ 7 Elementary symmetric functions

On (M^n, g) denote $A\colon = \mathrm{Ric} - \frac{R}{2(n-1)} g$, the *conformal Ricci tensor*; compare with Example 4 of Chapter 4. Then the full Riemannian curvature tensor Riem decomposes as

$$\text{Riem } = W + \frac{1}{n-2} A \owedge g,$$

where $\owedge$ denotes the Kulkarni–Nomizu product. Let h, k be two covectors and x_1, x_2, x_3, x_4 vectors, then

$$\begin{aligned}&(h \owedge k)(x_1,x_2,x_3,x_4)\\ &\colon = h(x_1,x_3)k(x_2,x_4) + h(x_2,x_4)k(x_1,x_3) - h(x_1,x_4)k(x_2,x_3) - h(x_2,x_3)k(x_1,x_4).\end{aligned}$$

The conformal Ricci tensor A is natural in conformal geometry. In his thesis J. Viaclovsky [90] considered the functional

$$\mathcal{F}_k(g)\colon = \int_M \sigma_k(A_g)\, dv_g,$$

where $\sigma_k(A)$ is the kth elementary symmetric function of the eigenvalues of the tensor A, e.g., if A is the conformal Ricci tensor,

$$\begin{aligned}
k = 1\colon\ & \sigma_1(A) = TrA = R - \frac{Rn}{2(n-1)} = \frac{n-2}{2(n-1)} R,\\
k = 2\colon\ & \sigma_2(A) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}[(TrA)^2 - |A|^2],\\
& \vdots\\
k = n\colon\ & \sigma_n(A) = \det A.
\end{aligned}$$

Theorem 7.1 [90] *If $k \neq \frac{n}{2}$ and if M is locally conformally flat, then*

$$\sigma_k(A_g) \equiv \text{const.}$$

for all metrics $g \in [g_0]$ that are critical for $\mathcal{F}_k[\cdot]$.

In this section, we are going to study $\sigma_2(A_g)$ on M^4. We remark that some of the algebraic properties of σ_2 on M^4 listed below have analogous forms for σ_k on M^n, see [48].

Denote

$$\begin{aligned}
A_{ij} &= R_{ij} - \frac{R}{2(n-1)} g_{ij} = R_{ij} - \frac{R}{6} g_{ij},\\
S_{ij} &= -E_{ij} + \frac{R}{4} g_{ij} = -R_{ij} + \frac{R}{2} g_{ij},\\
\sigma_2 &= \sigma_2(A) = \frac{1}{2}\left(\frac{1}{12} R^2 - |E|^2\right),
\end{aligned} \tag{7.1}$$

and recall that $R_{ij} = E_{ij} + \frac{R}{4} g_{ij}$.

Lemma 7.2

(a) *$R^2 \geq 24\sigma_2(A)$ with equality iff $E = 0$.*
In particular, if $\sigma_2(A) > 0$, then either $R > 0$ or $R < 0$ on M^4.

(b) *Let $S^{ij} \colon = g^{ik} g^{jl} S_{kl}, g \colon = e^{2w} g_0$, then*

$$\sigma_2(A_g) = \frac{1}{2} S^{ij} A_{ij} = \frac{1}{2} \langle S, A \rangle_g.$$

(c) *If $R > 0$ at $p \in M$, then for all $x \in T_p M$ and $S = S_{ij}$ one obtains*

$$\begin{aligned} S(x,x) &\geq \frac{3\sigma_2(A)}{R} g(x,x), \\ \mathrm{Ric}(x,x) &\geq \frac{3\sigma_2(A)}{R} g(x,x). \end{aligned}$$

Proof. (a) is immediate.

(b) Recall that the inner product of two 2-tensors h, k in the metric g is given by

$$\begin{aligned} \langle h, k \rangle_g &= g^{i\alpha} g^{j\beta} h_{ij} k_{\alpha\beta}, \\ S^{ij} A_{ij} &= \left(-E^{ij} + \frac{R}{4} g^{ij} \right) \left(E_{ij} + \frac{R}{12} g_{ij} \right) \\ &= -|E|^2 + \frac{R^2}{48} \cdot 4 = \frac{R^2}{12} - |E|^2 \\ &= 2\sigma_2(A), \end{aligned}$$

where we have used the property that $TrE = E^{ij} g_{ij} = 0$.

(c) Using Lemma 6.12 we estimate

$$|E(x,x)| \leq \frac{\sqrt{3}}{2} |E| |x|_g^2 \ \forall x \in T_p M.$$

Hence

$$\begin{aligned} S(x,x) &= -E(x,x) + \frac{R}{4} |x|_g^2 \\ &\geq \left(-\frac{\sqrt{3}}{2} |E| + \frac{R}{4} \right) |x|_g^2 \\ &\geq \left(-\frac{\sqrt{3}}{4} \left(c \frac{|E|^2}{R} + \frac{R}{c} \right) + \frac{R}{4} \right) |x|_g^2 \\ &= \left(-\frac{3}{2} \frac{|E|^2}{R} + \frac{1}{8} R \right) |x|_g^2 = \frac{3\sigma_2(A)}{R} |x|_g^2, \end{aligned}$$

if we choose $c \colon = 2\sqrt{3}$.

Similarly,

$$\mathrm{Ric}(x,x) = E(x,x) + \frac{R}{4} g(x,x) \geq \frac{3\sigma_2}{R} |x|_g^2 = \frac{3\sigma_2(A)}{R} g(x,x). \qquad \square$$

Corollary 7.3 (Corollary of (b) and (c) in Lemma 7.2) *If* $\sigma_2 = \sigma_2(A) > 0, R > 0$, *then*

$$\frac{R}{2} g_{ij} \underset{(b)}{\geq} R_{ij} \underset{(c)}{\geq} \frac{3\sigma_2}{R} g_{ij}.$$

In particular, Ric *is positive definite* ($R = c\sigma_1(A)$).

We now list some basic facts concerning the tensors S, A, and σ_2 etc. under conformal change of metrics. Let $g = g_w = e^{2w} g_0$, where g_0 is the background metric. Then

$$R = R_g = e^{-2w}(R_0 - 6\Delta_0 w - 6|\nabla_0 w|_0^2). \tag{7.2}$$

Notice the change of signs when using the g-metric instead of g_0. In fact,

$$\begin{aligned} R_0 &= e^{2w}(R + 6\Delta w - 6|\nabla w|^2) \\ \Rightarrow R &= e^{-2w} R_0 - 6\Delta w + 6|\nabla w|^2. \end{aligned} \tag{7.3}$$

Moreover,

$$\mathrm{Ric} = \mathrm{Ric}_0 - 2\nabla_0^2 w - (\Delta_0 w) g_0 + 2\, dw \otimes_0 dw - 2|\nabla_0 w|_0^2 g_0, \tag{7.4}$$

or in terms of g on the right-hand side:

$$\mathrm{Ric} = \mathrm{Ric}_0 - 2\nabla^2 w - (\Delta w) g - 2\, dw \otimes dw + 2|\nabla w|^2 g. \tag{7.5}$$

Analogously,

$$A = A_0 - 2\nabla_0^2 w + 2\, dw \otimes_0 dw - |\nabla_0 w|_0^2 g_0, \tag{7.6}$$

$$A = A_0 - 2\nabla^2 w - 2\, dw \otimes dw + |\nabla w|^2 g. \tag{7.7}$$

$$S = S_0 + 2\nabla_0^2 w - 2(\Delta_0 w) g_0 - 2\, dw \otimes_0 dw - |\nabla_0 w|_0^2 g_0, \tag{7.8}$$

$$S = S_0 + 2\nabla^2 w - 2(\Delta w) g + 2\, dw \otimes dw + |\nabla w|^2 g. \tag{7.9}$$

The behavior of $\sigma_2(A_g)$ under conformal change is determined by ($A = A_g$ for $g = e^{2w} g_0$)

$$\begin{aligned} \sigma_2(A) e^{4w} = \sigma_2(A_0) &+ 2\big[(\Delta_0 w)^2 - |\nabla_0^2 w|_0^2 \\ &+ \langle \nabla_0 w, \nabla_0(|\nabla_0 w|_0^2)\rangle_0 + \Delta_0 w |\nabla_0 w|_0^2\big] \\ &- 2(\mathrm{Ric})_0(\nabla_0 w, \nabla_0 w) - 2\langle S_0, \nabla_0^2 w\rangle_0. \end{aligned} \tag{7.10}$$

The last two terms are frequently denoted as lower-order terms. Notice that for $u \in C^\infty(M)$, one has

$$\sigma_2(\nabla_0^2 u) = \frac{1}{2}\big[(\Delta_0 u)^2 - |\nabla_0^2 u|_0^2\big],$$

which resembles the first two terms on the right-hand side of (7.10). $\sigma_2(\nabla_0^2 u)$ is a typical example of a fully nonlinear differential expression studied by Caffarelli, Nirenberg and Spruck [17] [18].

A fully non-linear differential equation of second order

$$\mathcal{F}(\nabla^2 u(x), \nabla u(x), u(x), x) = 0 \text{ in } \Omega \subset \mathbb{R}^n$$

is called *elliptic*, iff there are constants $0 < \theta_1 \leq \theta_2$, such that

$$\theta_1|\xi|^2 \leq \left(\frac{\partial \mathcal{F}}{\partial u_{ij}}\right) \xi_i \xi_j \leq \theta_2|\xi|^2$$

for all $\xi \in \mathbb{R}^n$.

In case $\mathcal{F}(\nabla^2 w, \nabla w, w, x) = \sigma_2(A_{g_w})$, one gets

$$\frac{\partial \mathcal{F}}{\partial w_{ij}} = -2S^{ij},$$

and if $\sigma_2(A_{g_w}) > 0$, then $(-\mathcal{F})$ is elliptic.

Lemma 7.4 (Divergence structure of σ_2) *For $\sigma_2(A) = \sigma_2(A_{g_w})$ one has*

(a) $\sigma_2(A)e^{4w} = \sigma_2(A_0) - \nabla_0(M(w)\nabla_0 w)$, *where*

$$M(w) := 2S_0 + 2\nabla_0^2 w - 2(\Delta_0 w)g_0 - 2\nabla_0 w \otimes \nabla_0 w, \tag{7.11}$$

(b) $M(w) = S + S_0 + |\nabla_0 w|_0^2 g_0$,

(c)
$$\nabla S = 0. \tag{7.12}$$

In particular, for M closed, compact,

$$\int_M S\nabla^2 f\, dv = -\int_M (\nabla S)\nabla f\, dv = 0 \quad \forall f \in C^2(M).$$

Proof. (a) follows from a straightforward computation from (7.10);
(b) follows from (7.8) and (7.11);
(c) follows from the first Bianchi identity

$$S_{ij} = -R_{ij} + \frac{R}{2}g_{ij} \Rightarrow \nabla_j S_{ij} = -\nabla_j R_{ij} + \frac{1}{2}\nabla_i R = 0. \qquad \square$$

The main theorem in [23] and [24] is

Theorem 7.5 *On (M^4, g_0) closed, compact, suppose*

(i) $Y(M, g_0) > 0$,

(ii) $\int_M \sigma_2(A_0)\, dv_0 > 0$.

Then there is $w \in C^\infty(M)$ with $\sigma_2(A_{g_w}) \equiv c > 0$.

Corollary 7.6 *Under the assumption of Theorem 7.5 there is $w \in C^\infty(M)$, with*

$$R_{g_w} > 0 \text{ and } (R_{g_w}/2) > (\mathrm{Ric})_{g_w} > 0.$$

Remark 7.7 The condition (ii) in Theorem 7.5 implies a topological constraint, which may be seen as follows. Assume that M^4 is orientable. According to the Chern–Gauss–Bonnet Theorem, one has

$$8\pi^2\chi(M^4) = \frac{1}{4}\int_M |W|^2\,dv + \int_M \sigma_2(A)\,dv. \tag{7.13}$$

In addition, the Signature Formula reads as

$$12\pi^2\tau(M^4) = \frac{1}{4}\left(\int_M [|W^+|^2 - |W^-|^2]\right)\,dv \tag{7.14}$$

where

$$\begin{aligned} W^+ &:= \text{self-dual part of } W,\\ W^- &:= \text{anti-self-dual part of } W,\\ \tau &:= \text{signature of } M^4 \text{ (a topological invariant).}\end{aligned}$$

Adding (7.13) and (7.14) we arrive at

$$4\pi^2(2\chi(M^4) \pm 3\tau(M^4)) = \frac{1}{2}\int_M |W^\pm|^2\,dv + \int_M \sigma_2(A)\,dv.$$

Thus (ii) in Theorem 7.5 implies the constraint

$$2\chi(M^4) \pm 3\tau(M^4) > 0. \tag{7.15}$$

Examples. For simply connected 4-manifolds with positive scalar curvature, there is a well-known work of Donaldson [40] see also [47] that up to homeomorphism type, the manifolds are

$$k(\mathbb{CP}^2)\#l(\overline{\mathbb{CP}^2}) \text{ or } k(S^2 \times S^2).$$

If we assume in addition that $\int \sigma_2(A_g)dv_g > 0$, then Condition (7.15) implies

$$0 < k < 4 + 5l, \tag{7.16}$$

where $\chi = k + l + 2, \tau = k - l$, e.g. for $l = 0, k < 4$. We remark that for manifolds of this type Sha–Yang [83] have already shown the existence of a metric $\tilde{g}$ with $(\mathrm{Ric})_{\tilde{g}} > 0$.

Remark 7.8 To prove Theorem 7.5 we will proceed in two steps. First we deform the given background metric g_0 in the conformal class to some metric g_w with $\sigma_2(A_{g_w}) = f > 0$ for some positive function f. Secondly, we will deform f to be constant. To be more precise, we will first show

Theorem 7.9 *Under the assumption of Theorem* 7.5 *there is $f \in C^\infty(M), f > 0$ and $w \in C^\infty(M)$ such that $\sigma_2(A_{g_w}) = f > 0$.*

The second step will be the proof of

Theorem 7.10 *Suppose there is $w \in C^\infty(M)$, such that*

(i)′ $R_{g_w} > 0$
(ii)′ $\sigma_2(A_{g_w}) = f > 0$ *for some* $f \in C^\infty(M)$.

If (M^4, g) is not conformally equivalent to (S^4, g_c), then there exists a constant

$$C_1 = C_1\left(||f||_{C^1}, \left(\min_M f(\cdot)\right)^{-1}, g\right)$$

such that

$$||w||_{L^\infty} \leq C_1.$$

We have to exclude the case of conformal equivalence to (S^4, g_c), since, for instance, on (S^4, g_c), if $e^{2w} g_c = \phi^*(g_c)$, then one has in Euclidean coordinates,

$$w_\lambda(x) = \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2}$$

and $\sigma_2(A_{g_{w_\lambda}}) \equiv 6$ for all $\lambda > 0$, but

$$\lim_{\lambda \to 0} ||w_\lambda||_{L^\infty} = \infty.$$

Once Theorem 7.10 is shown we will be able to conclude that there is a constant $C_2 = C_2(||f||_{C^\infty}, C_1)$ with $||w||_{C^\infty} \leq C_2$.

By means of degree theory we finally prove

Corollary 7.11 *If (M^4, g_0) is a closed compact 4-manifold satisfying* (i), (ii) *of Theorem* 7.5, *then there is $w \in C^\infty(M)$, such that*

$$\sigma_2(A_{g_w}) \equiv 1.$$

We will prove Theorem 7.9 in Chapters 8 and 9; and Theorem 7.10 in Chapter 10.

§ 8 A priori estimates for the regularized equation $(*)_\delta$

In this chapter we will prove Theorem 7.9.

Theorem 8.1 [23] *On (M^4, g_0) closed, compact, assume*
(i) $Y(M^4, g_0) > 0$,
(ii) $\int_M \sigma_2(A_0)\, dv_0 > 0$;
then there is $f \in C^\infty(M), f > 0$, and $w \in C^\infty(M)$, such that

$$\sigma_2(A_{g_w}) = f.$$

Remark. Conditions (i) and (ii) are invariant under conformal change of the metric, so sometimes we will simply write $Y(M)$ or $\int_M \sigma_2(A)\, dv$ without specifying the metric.

Outline of the proof. We will use a continuity method on the "regularized equation" (in terms of $g = e^{2w} g_0$)

$$\delta \Delta R = 8\gamma_1 |W|^2 + 4\sigma_2(A). \tag{$(*)_\delta$}$$

As we take the formal limit $\delta \to 0$ we end up with

$$f = -2\gamma_1 |W|^2.$$

To make sure that f thus found is positive, we first observe that under the assumption (ii) of Theorem 8.1, $\gamma_1 < 0$. Thus $f \geq 0$. Later on we will modify f to get $f > 0$ at points where the norm of the Weyl tensor $|W| = 0$.

There will be two main steps in the proof of Theorem 8.1

Step 1. For all $\delta > 0$ there is $w \in C^\infty(M)$ solving $(*)_\delta$ with $R = R_{g_w} > 0$.

Step 2. We will show a priori estimates for solutions of $(*)_\delta$ independent of δ as $\delta \to 0$.

Before setting up Step 1 notice that solving $(*)_\delta$ amounts to analytically solving

$$-6\delta \Delta^2 w = 8((\Delta w)^2 - |\nabla^2 w|^2 + \cdots) - 4f.$$

Step 1. Fix $\delta_0 > 0$, and consider the set

$$\mathcal{S} \colon = \{\delta \in [\delta_0, 1] : (*)_\delta \text{ admits a smooth solution } w \text{ with } R_{g_w} > 0\}.$$

Lemma 8.2 *Under the hypotheses (i), (ii) of Theorem* 8.1, *one finds* $1 \in \mathcal{S}$, *i.e.,* $\mathcal{S} \neq \emptyset$.

Proof. Apply Theorem 6 with the choice $\gamma_2 = 1, \gamma_3 = \frac{1}{24}(3\delta - 2) = \frac{1}{24}$, and $\gamma_1 \leq 0$, such that $k_d = 0$; compare with Chapter 6.

We find a solution $w \in C^\infty(M)$ with

$$\begin{aligned}\Delta R &= 8\gamma_1|W|^2 + 4\left(\frac{1}{24}R^2 - \frac{1}{2}|E|^2\right)\\ &= 8\gamma_1|W|^2 + \frac{1}{6}R^2 - 2|E|^2\\ &\le \frac{1}{6}R^2,\end{aligned}$$

all in terms of the metric $g = e^{2w}g_0$.

The last inequality means $LR \geq 0$, which implies by Lemma 6.8 and hypothesis (ii) that $R > 0$, hence $1 \in \mathcal{S}$. □

Lemma 8.3 *$\mathcal{S}$ is open.*

Proof. If $\delta_1 \in \mathcal{S}, g_1 : = e^{2w_1}g, R_{g_1} > 0$, then we know from Proposition 6.14, that $\ker \mathcal{L}_{\delta_1} = \mathbb{R}$, where $\mathcal{L}_{\delta_1}$ is the linearization of $(*)_{\delta_1}$. According to [2] one finds for every δ sufficiently close to δ_1 a smooth solution $w_\delta \in C^\infty(M)$ of $(*)_\delta$. Since $R_{g_1} > 0$ we get $R_{g_w} > 0$ for all w sufficiently close to w_1 in the $C^{2,\alpha}$-norm, i.e., $R_{g_{w_\delta}} > 0$ for all δ sufficiently close to δ_1. □

Lemma 8.4 *$\mathcal{S}$ is closed.*

Proof. Our aim is to show that for $\delta_k \in \mathcal{S}$ with $\delta_k \to \bar{\delta}$ with $\bar{\delta} \geq \delta_0 > 0$, we find that a subsequence of the w_{δ_k} converges to a solution $w_{\bar{\delta}}$ of $(*)_{\bar{\delta}}$ in $W^{2,2}(M^4)$. The result in [89] implies that $w_{\bar{\delta}} \in C^\infty(M)$. Thus Lemma 8.4 follows directly from the following a priori estimates, in particular from (8.2). □

Proposition 8.5 *Suppose w with $g = g_w = e^{2w}g_0$ solves $(*)_\delta$ with $R = R_{g_w} > 0$. Assume that $\int_M w\, dv_0 = 0$, then there are constants C_0, C_1 depending only on the background metric g_0, such that*

$$w \geq C_0, \tag{8.1}$$

$$\delta \int_M (\Delta_0 w)^2\, dv_0 + \frac{2}{3}\int_M |\nabla_0 w|_0^4\, dv_0 \leq C_1. \tag{8.2}$$

Moreover, for any $\alpha \in \mathbb{R}, p \geq 0$, there are constants $C_2(\alpha, g), C_3(p, g)$, such that

$$\int_M e^{\alpha w}\, dv_0 \leq C_2, \tag{8.3}$$

$$\int_M |\nabla_0 w|_0^4 |w|^p\, dv_0 \leq C_p. \tag{8.4}$$

Proof. To prove (8.1) recall

$$\Delta_0 w + |\nabla_0 w|_0^2 + \frac{1}{6}R_{g_w}e^{2w} = \frac{1}{6}R_0, \tag{8.5}$$

which implies, by $R_{g_w} > 0$,

$$\Delta_0 w + |\nabla_0 w|_0^2 \leq \frac{1}{6} R_0, \tag{8.6}$$

in particular,

$$\Delta_0 w \leq \frac{1}{6} R_0. \tag{8.7}$$

Let $G(\cdot,\cdot)$ denote the Green's function of the operator Δ_0 on (M, g_0); then we may write according to Green's formula,

$$-w(x) + \int_M w \, dv_0 = \int_M G(x,y)(\Delta_0 w)(y) \, dv_0(y).$$

Since M is compact and closed, we may add a constant to G to get G positive. Then, if $\int_M w \, dv_0 = 0$ as we assumed, we obtain

$$w(x) \geq -\int_M G(x,y) \frac{R_0(y)}{6} \, dv_0(y) =: C_0.$$

To prove (8.2), we first integrate (8.6) over M to obtain

$$\int_M |\nabla_0 w|_0^2 \, dv_0 \leq \frac{1}{6} \int_M R_0 \, dv_0 =: \tilde{C}_1, \tag{8.8}$$

hence, by Poincaré's inequality,

$$\int_M w^2 \, dv_0 \leq \hat{C}_1, \tag{8.9}$$

since $\int_M w \, dv_0 = 0$. Now (8.2) follows from the weak form of the Euler–Lagrange equation $(*)_\delta$ in terms of analytic expressions in w. More precisely, for all $\varphi \in W^{2,2}(M)$,

$$\begin{aligned}
\int_M \Big(\frac{2}{3} \delta \Delta_0 w \Delta_0 \varphi &+ \frac{1}{2}(3\delta - 2)[\Delta_0 \varphi |\nabla_0 w|_0^2 + 2\Delta_0 w \langle \nabla_0 \varphi, \nabla_0 w \rangle_0 \\
&+ 2|\nabla_0 w|_0^2 \langle \nabla_0 \varphi, \nabla_0 w \rangle_0] \Big) \, dv_0 \qquad (8.10)\\
= \int_M \Big(-2U_0^\delta \varphi &+ 2 \operatorname{Ric}_0(\nabla_0 \varphi, \nabla_0 w) + \frac{1}{2}(\delta - 2) R_0 \langle \nabla_0 \varphi, \nabla_0 w \rangle \Big) \, dv_0,
\end{aligned}$$

where $U_0^\delta :\, = \gamma_1 |W_0|_0^2 + \gamma_2 Q_0 - \gamma_3 \Delta_0 R_0, \gamma_2 = 1, \gamma_3 = \frac{1}{24}(3\delta - 2)$, and $\gamma_1 \leq 0$ appropriately chosen, so that $k_d = 0$.

Notice that the right-hand side is of lower order and bounded according to (8.8) and (8.9). Testing with $\varphi :\, = w$ in (8.10) we get

$$\int_M \Big(\frac{3}{2} \delta (\Delta_0 w)^2 + \frac{3}{2}(3\delta - 2) \Delta_0 w |\nabla_0 w|_0^2 + (3\delta - 2)|\nabla_0 w|_0^4 \Big) \, dv_0 \leq C \tag{8.11}$$

for some constant C. (We will repeatedly use the notation C for generic constants, whose values might change from line to line in the following.)

Case 1. If $\delta \in \left[\frac{2}{3}, 1\right]$, i.e., $3\delta - 2 \in [0,1]$, we use $\frac{3}{2}xy \geq -\frac{9}{16}x^2 - y^2$ to obtain from, (8.11) for $x \colon = \Delta_0 w, y \colon = |\nabla_0 w|_0^2$,

$$\begin{aligned}\int_M \frac{3}{16}(6-\delta)(\Delta_0 w)^2\, dv_0 &= \int_M \left(\frac{3}{2}\delta - \frac{9}{16}(3\delta-2)\right)(\Delta_0 w)^2\, dv_0 \\ &\leq \int_M \left(\frac{3}{2}\delta(\Delta_0 w)^2 + \frac{3}{2}(3\delta-2)\Delta_0 w|\nabla_0 w|_0^2 + (3\delta-2)|\nabla_0 w|_0^4\right) dv_0 \\ &\leq C,\end{aligned}$$

i.e.,

$$\int_M (\Delta_0 w)^2\, dv_0 \leq C. \tag{8.12}$$

Notice also that by (8.6),

$$\begin{aligned}\int_M |\nabla_0 w|_0^4\, dv_0 &\leq \frac{1}{6}\int_M R_0|\nabla_0 w|_0^2\, dv_0 - \int_M (\Delta_0 w)|\nabla_0 w|_0^2\, dv_0 \\ &\leq \frac{1}{6\varepsilon}\int_M R_0^2\, dv_0 + \frac{1}{\varepsilon}\int_M |\Delta_0 w|^2\, dv_0 + 2\varepsilon\int_M |\nabla_0 w|_0^4\, dv_0,\end{aligned}$$

hence, by (8.12),

$$\int_M |\nabla_0 w|_0^4\, dv_0 \leq C,$$

which finishes the proof of (8.2) in Case 1.

Case 2. If $\delta \in \left(0, \frac{2}{3}\right)$, i.e., $(3\delta-2) \in (-2,0)$, then by (8.6),

$$\begin{aligned}(3\delta-2)\left[\frac{3}{2}\Delta_0 w + |\nabla_0 w|_0^2\right] &= (3\delta-2)\left[\frac{3}{2}(\Delta_0 w + |\nabla_0 w|_0^2) - \frac{1}{2}|\nabla_0 w|_0^2\right] \\ &\underset{(8.6)}{\geq} \frac{(3\delta-2)}{6}\cdot\frac{3}{2}R_0 + \frac{2-3\delta}{2}|\nabla_0 w|_0^2.\end{aligned}$$

Inserting this into (8.11) we obtain

$$\begin{aligned}&\frac{3}{2}\delta\int_M (\Delta_0 w)^2 dv_0 + \frac{1}{2}\int_M |\nabla_0 w|_0^4(2-3\delta)\, dv_0 \\ &\leq \int_M \frac{3}{2}\delta(\Delta_0 w)^2 dv_0 + \int_M \left((3\delta-2)\left[\frac{3}{2}\Delta_0 w + |\nabla_0 w|_0^2\right]|\nabla_0 w|_0^2 + \frac{(2-3\delta)}{6}\frac{3}{2}R_0\right) dv_0 \\ &\underset{(8.11)}{\leq} C,\end{aligned}$$

i.e.,

$$\frac{3}{2}\delta\int_M(\Delta_0 w)^2\,dv_0+\int_M|\nabla_0 w|_0^4\,dv_0-\frac{3}{2}\delta\int_M|\nabla_0 w|_0^4\,dv_0\le C. \tag{8.13}$$

On the other hand, multiplying (8.11) by $\frac{3\delta}{2}(2-3\delta)^{-1}>0$ leads to the estimate

$$\begin{aligned}-\frac{3}{2}\delta\int_M|\nabla_0 w|_0^4\,dv_0 &\le C+\frac{9\delta}{4}\int_M(\Delta_0 w)|\nabla_0 w|_0^2\,dv_0\\ &\le C+\frac{9\delta}{4}\left[\frac{1}{2}\int_M(\Delta_0 w)^2\,dv_0+\frac{1}{2}\int_M|\nabla_0 w|_0^4\,dv_0\right].\end{aligned}$$

Substituting this into (8.13) we get

$$\frac{3}{8}\delta\int_M(\Delta_0 w)^2\,dv_0+\left(1-\frac{9}{8}\delta\right)\int_M|\nabla_0 w|_0^4\,dv_0\le C,$$

or

$$\delta\int_M(\Delta_0 w)^2\,dv_0+\left(\frac{8}{3}-3\delta\right)\int_M|\nabla_0 w|_0^4\,dv_0\le C,$$

which proves (8.2), since $\delta\in\left(0,\frac{2}{3}\right)$ in this case. (8.3) follows from Adams' inequality, Lemmas 6.1 and 6.2 in the same way as Corollary 1.7 was deduced from Corollary 1.6. Notice that (8.2) guarantees that the constant on the right-hand side of (8.3) does not depend on w.

Testing (8.10) with $\varphi:=w^p$ and integrating by parts leads to (8.4); for details, see [23]. □

With Lemma 8.4 we have established the existence of smooth solutions w of $(*)_\delta$ with $R_{g_w}>0$ for all $\delta>0$. The following two results summarize the necessary a priori estimates independent of δ, as $\delta\to 0$.

Proposition 8.6 *Under the assumptions of Theorem* 8.1 *there is a constant* $C_1=C_1(g)$ *independent of* δ*, such that for the solutions* $w_\delta\in C^\infty(M)$ *of* $(*)_\delta$*,*

$$||w_\delta||_{W^{2,3}}\le C_1\ \ \forall\,\delta>0.$$

Proposition 8.7 *For all* $s<5$ *there is a constant* $C_2=C_2(g,s)$ *independent of* δ*, such that*

$$||w_\delta||_{W^{2,s}}\le C_2\ \ \forall\,\delta>0.$$

Before proving these a priori estimates let us review some regularity theory for fully nonlinear elliptic equations. The techniques used in [17], [18], [42], [60] motivate the approach we will present in these lectures.

The investigations in [17], [18] are concerned with the fully nonlinear elliptic equations of the form

$$\begin{cases} \mathcal{F}(\nabla^2 u, \nabla u, u, x) & = \varphi(x) \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u(x) & = \psi(x) \quad \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{F}$ is assumed to be uniformly elliptic, see Chapter 7. In [17] the Monge–Ampère equation ($\mathcal{F} = \det(u_{ij})$) is studied, whereas [18] includes the case $\mathcal{F} = \sigma_k(u_{ij})$. Omitting their results regarding boundary estimates, we will focus on *interior estimates* for $\mathcal{F}_k = \sigma_k(u_{ij})$.

Definition 8.8 $\Gamma_k^+ : \ = \{A \in M(n \times n)$ *with* $\sigma_k(A) > 0$ *and* A *is in the same connected component as the identity* $\}$.

Γ_k^+ is a convex cone with the following properties.

Proposition 8.9

(i) $\Gamma_k^+ \subseteq \Gamma_{k-1}^+ \subseteq \cdots \subseteq \Gamma_1^+$,

(ii) *For* $(u_{ij}) \in \Gamma_k^+, \sigma_k^{\frac{1}{k}}(u_{ij})$ *is a concave function, i.e., for* $A = (u_{ij}) \in \Gamma_k^+$ *and* $B = (v_{ij}) \in \Gamma_k^+$ *one has* $\sigma_k^{\frac{1}{k}}(tA + (1-t)B) \geq t\sigma_k^{\frac{1}{k}}(A) + (1-t)\sigma_k^{\frac{1}{k}}(B)$,

(iii) *Let* $(u_{ij}) \in \Gamma_k^+$ *with* $\mathcal{F}_k(u_{ij}) = \sigma_k^{\frac{1}{k}}(u_{ij}) = \varphi$ *for some given smooth function* φ *with*

$$0 < \inf_\Omega \varphi \leq \varphi \leq \sup_\Omega \varphi < \infty,$$

then $u \in C^0(\Omega) \Rightarrow u \in C^1(\Omega) \Rightarrow u \in C^2(\Omega) \Rightarrow u \in C^{2,\alpha}(\Omega), \Rightarrow u \in C^\infty(\Omega)$, *with the interior estimates*

$$\begin{aligned} ||u||_{C^1(B_R)} &\lesssim ||u||_{C^0(B_{2R})}, \\ ||u||_{C^2(B_R)} &\lesssim ||u||_{C^1(B_{2R})}, \\ ||u||_{C^{2,\alpha}(B_R)} &\lesssim ||u||_{C^2(B_{2R})}, \\ ||u||_{C^\infty(B_R)} &\lesssim ||u||_{C^{2,\alpha}(B_{2R})}, \end{aligned}$$

where $\lesssim$ *denotes the inequality up to a constant factor depending on the data, in particular on* φ.

(iv) $u \in C^{1,1}(\Omega) \Rightarrow u \in C^{2,\alpha}(\Omega)$ *if* $\mathcal{F}_{\|}$ *is uniformly elliptic and concave, see* [42], [60].

To motivate our method to establish a priori bounds in $W^{2,3}$, we will first establish an a priori estimate for solutions w of the equation $\sigma_2(A_{g_w}) = f > 0$ on M^4.

Theorem 8.10 *Let* $w \in C^\infty(M^4), (M^4, g_0)$ *closed, compact, satisfy* $\sigma_2(A_{g_w}) = f$, *for some* $f > 0$ *on* M^4, *with* $R_{g_w} > 0$. *Then*

$$||\nabla_0^2 w||_{L^\infty} \leq C(g_0, \min_M f(\cdot), ||w||_{L^\infty}, ||\nabla_0 w||_{L^\infty} ||f||_{C^3}).$$

The outline of the proof of Theorem 8.10 is as follows. Recall from Lemma 7.2 that the linearization of σ_2 is essentially given by the tensor $S = (S_{ij})$, for which we derive an identity involving the *Bach tensor* $B = (B_{ij})$ in Lemma 8.11. To prepare a variant of Pogorelov's trick we analyze the expression $S^{ij}\nabla_i\nabla_j V$ for $V\colon = \frac{1}{2}|\nabla w|^2$ in Lemma 8.13, before we apply the maximum principle.

Lemma 8.11 *Calculating in the metric $g_w = e^{-2w}g$,*

$$\begin{aligned} S^{ij}\nabla_i\nabla_j R = {} & 3\Delta\sigma_2(A) + 3\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) \\ & + 6TrE^3 + R|E|^2 \\ & - 6W^{ijkl}E_{ik}E_{jl} - 6E^{ij}B_{ij}, \end{aligned} \tag{8.14}$$

where B_{ij} denotes the Bach tensor, which is the first variation of $\int_M |W|^2$, given by

$$B_{ij} = \nabla^k\nabla^l W_{kijl} + \frac{1}{2}R^{kl}W_{kijl}.$$

Notice that the only property relevant for us is the behavior of $B = (B_{ij})$ under conformal change of the metric:

$$B = B_{g_w} = e^{-2w}B_0.$$

Proof of Lemma 8.11. Applying the Bianchi identity, by a formulation of Derdzinski [39] we have

$$\begin{aligned} B_{ij} = & -\frac{1}{2}\Delta E_{ij} + \frac{1}{6}\nabla_i\nabla_j R - \frac{1}{24}\Delta R g_{ij} \\ & - E^{kl}W_{ikjl} + E_i^k E_{jk} - \frac{1}{4}|E|^2 g_{ij} + \frac{1}{6}RE_{ij}, \end{aligned} \tag{8.15}$$

where $E_i^k\colon = g^{k\alpha}E_{\alpha i}$.

Thus

$$\begin{aligned} \frac{1}{2}\Delta|E|^2 &= |\nabla E|^2 + E^{ij}\Delta E_{ij} \\ &\underset{(8.15)}{=} |\nabla E|^2 + \frac{1}{3}E^{ij}\nabla_i\nabla_j R + 2TrE^3 + \frac{1}{3}R|E|^2 - 2W^{ikjl}E_{ij}E_{kl} - 2B^{ij}E_{ij}, \end{aligned}$$

where we used the fact that $TrE = E^{ij}g_{ij} = 0$.

Consequently,

$$\begin{aligned} \Delta\sigma_2(A) &= \Delta\left(-\frac{1}{2}|E|^2 + \frac{1}{24}R^2\right) \\ &= -|\nabla E|^2 + \frac{1}{12}|\nabla R|^2 + \frac{1}{12}R\Delta R - \frac{1}{3}E^{ij}\nabla_i\nabla_j R \\ &\quad - 2TrE^3 - \frac{1}{3}R|E|^2 + 2W^{ikjl}E_{ij}E_{kl} + 3B^{ij}E_{ij}. \end{aligned}$$

Note that $\frac{1}{12}R\Delta R - \frac{1}{3}E^{ij}\nabla_i\nabla_j R = \frac{1}{3}S^{ij}\nabla_i\nabla_j R$, by definition of $S = (S_{ij})$, see Chapter 7, which proves (8.14). □

We now begin the proof of Theorem 8.1

Notice that for $\sigma_2 = \sigma_2(A) = f > 0$ with $R > 0$ we can argue as follows:

$$\begin{aligned}\nabla\sigma_2 &= \frac{1}{12}R\nabla R - |E|\nabla(|E|), \quad \text{i.e.,}\\ \left\langle -\nabla\sigma_2, \frac{\nabla R}{R}\right\rangle &= -\frac{1}{12}|\nabla R|^2 + \frac{|E|}{R}\langle\nabla|E|, \nabla R\rangle\\ &\leq \frac{1}{2}\frac{|E|^2}{R^2}|\nabla R|^2 + \frac{1}{2}|\nabla(|E|)|^2 - \frac{1}{12}|\nabla R|^2\\ &\leq \frac{1}{2}|\nabla E|^2 + \frac{|\nabla R|^2}{R^2}\left(\frac{1}{2}|E|^2 - \frac{1}{24}R^2 + \frac{1}{24}R^2\right) - \frac{1}{12}|\nabla R|^2\\ &\leq \frac{1}{2}\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) - \sigma_2\frac{|\nabla R|^2}{R^2},\end{aligned}$$

where we used Kato's inequality, $|\nabla(|E|)| \leq |\nabla E|$.

Thus

$$\frac{1}{2}\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) \geq \sigma_2\frac{|\nabla R|^2}{R^2} - \nabla\sigma_2\frac{\nabla R}{R}. \tag{8.16}$$

At a point $p \in M$ with $R(p) = \max_M R$ one has $\nabla R = 0$ and $S^{ij}\nabla_i\nabla_j R \leq 0$, since S_{ij} is positive definite according to Lemma 7.2 (c). Since E is traceless,

$$\begin{aligned}6TrE^3 + R|E|^2 &\geq -\frac{6}{\sqrt{3}}|E|^3 + R|E|^2\\ &\geq |E|^2(R - 2\sqrt{3}|E|)\\ &= |E|^2\frac{R^2 - 12|E|^2}{R + 2\sqrt{3}|E|} = |E|^2\frac{24\sigma_2}{R + 2\sqrt{3}|E|}\\ &\geq |E|^2\frac{12\sigma_2}{R} > 0,\end{aligned} \tag{8.17}$$

because $\sigma_2 > 0$ implies $\frac{1}{12}R^2 > |E|^2$, i.e., $2\sqrt{3}|E| < R$.

Furthermore,

$$|WEE| \leq e^{-2w}|W_0|_0|E|^2 \lesssim |E|^2, \tag{8.18}$$

under the assumptions that $||w||_{L^\infty}$ and $|W_0|_0$ are controlled.

Similarly,

$$|BE| \leq e^{-2w}|B_0|_0|E| \lesssim |E|, \tag{8.19}$$

where again $\lesssim$ denotes an inequality up to a multiplicative constant.

Combining (8.16) and (8.17) we obtain

$$\begin{aligned}S^{ij}\nabla_i\nabla_j R \geq 3\Delta\sigma_2 &+ 6\left(\sigma_2\frac{|\nabla R|^2}{R^2} - \nabla\sigma_2\frac{\nabla R}{R}\right)\\ &+ |E|^2\frac{12\sigma_2}{R} + WEE + BE,\end{aligned} \tag{8.20}$$

and at a maximum point $p \in M$ of $R(\cdot)$ we have

$$0 \geq (S^{ij}\nabla_i\nabla_j R)(p) \geq 3\Delta\sigma_2(p) + |E|^2 \frac{12\sigma_2}{R}(p) - C_1|E|^2(p) - C_2|E|(p).$$

But it is not clear, if the right-hand side dominates some term like $cR^2 - cR$. The estimate (8.20), however, is still useful to prove the following uniqueness result.

Corollary 8.12 ([90]) *If $\sigma_2(A_{g_w}) \equiv$ const. $=: c > 0$, for the metric $g_w = e^{2w}g_c$ on S^4, then $R_{g_w} \equiv$ const., and $g_w = \phi^*(g_c)$ for some conformal transformation $\phi\colon S^4 \to S^4$.*

Proof. On (S^4, g_c) one has $(W_{ijkl})_{g_w} \equiv 0$ for $g_w \in [g_c]$, and therefore also $B_{g_w} \equiv 0$, and (8.20) simplifies to

$$\begin{aligned} S^{ij}\nabla_i\nabla_j R &\geq 6c\frac{|\nabla R|^2}{R} + |E|^2\frac{12c}{R} \\ &\geq 6c\frac{|\nabla R|^2}{R}. \end{aligned}$$

By (7.12) in Lemma 7.4, we obtain

$$0 = \int_{S^4} S^{ij}\nabla_i\nabla_j R\, dv_{g_w} \geq 6c\int_{S^4} \frac{|\nabla R|^2}{R}\, dv_{g_w},$$

i.e., $R = R_{g_w} \equiv$ const., which by Obata's Theorem implies $g_w = \phi^*(g_c)$. □

To make use of (8.20) for the proof of Theorem 8.10 we use Pogorelov's trick [76] applying the maximum principle to a function of the type $(\Delta w)e^{\varphi(|\nabla w|^2)}$ for some suitably chosen function φ.

Lemma 8.13 *On (M^4, g_w) let $V\colon = \frac{1}{2}|\nabla_{g_w} w|^2_{g_w} =: \frac{1}{2}|\nabla w|^2$.*
Then, in terms of the metric g_w,

$$\begin{aligned} S^{ij}\nabla_i\nabla_j V = &-\frac{1}{4}TrE^3 + \frac{1}{48}R|E|^2 + \frac{1}{(24)^2}R^3 \\ &-\frac{1}{2}\langle\nabla w, \nabla\sigma_2\rangle + \textit{ lower order terms} \\ &\textit{of order}(|\nabla w|^2|\nabla^2 w|^2, |\nabla^2 w|^2, |\nabla w|^6, \textit{ etc.}). \end{aligned} \tag{8.21}$$

Proof. With respect to the metric g_w we compute the covariant derivatives of V first:

$$\begin{aligned} \nabla_j V = \nabla_j\left(\frac{1}{2}|\nabla w|^2\right) &= \nabla_j(\nabla_k w\nabla_k w), \\ \nabla_i\nabla_j V &= (\nabla_i\nabla_k w)(\nabla_j\nabla_k w) + \langle\nabla_i\nabla_j\nabla_k w)\nabla_k w, \\ \nabla_i\nabla_j\nabla_k w &= \nabla_i\nabla_k\nabla_j w = \nabla_k\nabla_i\nabla_j w + R^m_{ikj}\nabla_m w. \end{aligned}$$

Recall (7.6),

$$\nabla_i\nabla_j w = -\frac{1}{2}A_{ij} + \frac{1}{2}A^0_{ij} - \nabla_i w\nabla_j w + \frac{1}{2}|\nabla w|^2(g_w)_{ij}. \tag{8.22}$$

So,

$$\nabla_i\nabla_j V = \nabla_i\nabla_k w\nabla_j\nabla_k w - \frac{1}{2}\nabla_k A_{ij}\nabla_k w + \text{ l.o.t. of order } (|\nabla^2 w|\cdot|\nabla w|^2).$$

Thus

$$S^{ij}\nabla_i\nabla_j V = S^{ij}\nabla_i\nabla_k w\nabla_j\nabla_k w - \frac{1}{2}S^{ij}\nabla_k w(\nabla_k A_{ij}) + \text{ l.o.t.of order } (|\nabla^2 w|\cdot|\nabla w|^2). \tag{8.23}$$

Notice that by (8.22) and (7.1)

$$\begin{aligned} S^{ij}\nabla_i\nabla_k w\nabla_j\nabla_k w &\underset{(8.22)}{=} \frac{1}{4}S^{ij}A_{ik}A_{jk} \\ &\quad + \text{ l.o.t. of order } (|\nabla^2 w|^2|\nabla w|^2, |\nabla w|^4) \\ &\underset{(7.1)}{=} -\frac{1}{4}TrE^3 + \frac{R}{48}|E|^2 + \frac{1}{576}R^3 \\ &\quad + \text{ l.o.t. of order } (|\nabla^2 w|^2|\nabla w|^2|\nabla w|^4). \end{aligned} \tag{8.24}$$

Moreover

$$S^{ij}\nabla_k w\nabla_k A_{ij} = \langle\nabla w, \nabla\sigma_2(A)\rangle, \tag{8.25}$$

since by (7.1),

$$\begin{aligned} (\nabla_k A^{ij})S^{ij} &= \left(\nabla_k E^{ij} + \frac{1}{12}(\nabla_k R)g^{ij}\right)\left(-E_{ij} + \frac{1}{4}Rg_{ij}\right) \\ &= -E^{ij}(\nabla_k E_{ij}) + \frac{1}{12}R\nabla_k R \\ &= \nabla_k\left(-\frac{1}{2}|E|^2 + \frac{1}{24}R^2\right) \\ &= \nabla_k\sigma_2. \end{aligned}$$

Summarizing (8.23)–(8.25) completes the proof. □

Proof of Theorem 8.10. We calculate in terms of the metric $g_w = e^{2w}g_0$. First notice by $\sigma_2 = \sigma_2(A_{g_w}) = f > 0$, that $S \geq \frac{3\sigma_2}{R} > 0$ by Lemma 7.2 (c). In addition, for $|\nabla w| \leq c, |w| \leq c$, one gets

$$\begin{aligned} &|E|^2 \leq 12R^2 + C(f), \text{ i.e.,} \\ &|\text{Ric}|^2 \lesssim R^2 + C, \text{ or in terms of } w, \\ &|\nabla^2 w| \lesssim |\Delta w| \lesssim |\nabla^2 w|. \end{aligned}$$

We apply the maximum principle to the function $h\colon = R + 24V$. At a maximum point $p \in M$ of h we have, by Lemmas 8.11 and 8.13,

$$
\begin{aligned}
0 \geq S^{ij}(p)\nabla_i\nabla_j h(p) &= S^{ij}(p)\nabla_i\nabla_j R(p) + 24 S^{ij}(p)\nabla_i\nabla_j V(p) \\
&= 3\Delta\sigma_2(p) + 3\left(|\nabla E|^2(p) - \frac{1}{2}|\nabla R|^2(p)\right) \\
&\quad + \frac{3}{2}R(p)|E|^2(p) + \frac{1}{24}R^3(p) \\
&\quad - 12\langle \nabla w(p), \nabla\sigma_2(p)\rangle \\
&\quad + \text{ l.o.t. of order } (|\nabla^2 w|^2|\nabla w|^2).
\end{aligned}
$$

Now use (8.16) to estimate the term in brackets to get (by $|\nabla w| \leq c$),

$$
\begin{aligned}
0 \geq S^{ij}(p)\nabla_i\nabla_j h(p) &\gtrsim \frac{1}{24}R^3(p) + \frac{3}{2}R(p)|E|^2(p) \\
&\quad - c(||f||_{C^2}) - c(||f||_{C^1})\left|\frac{\nabla R}{R}\right|(p) - cR^2 - c.
\end{aligned}
$$

At p we have $\nabla h(p) = 0$, thus

$$
|\nabla R|(p) = 24|\nabla V|(p) \lesssim |\nabla^2 w(p)||\nabla w(p)|,
$$

and $\sigma_2(p) \geq \min_M f(\cdot) > 0$, which implies

$$
R(p) \gtrsim \left(\min_M f(\cdot)\right)^{\frac{1}{2}} > 0,
$$

so

$$
\left|\frac{\nabla R}{R}\right|(p) \lesssim |\nabla^2 w(p)||\nabla w(p)| \lesssim |\nabla^2 w(p)|.
$$

Consequently, there exist constants c_1, c_2, c_3 depending on $(f, |\nabla w|, |w|)$, such that

$$
0 \geq S^{ij}(p)\nabla_i\nabla_j h(p) > c_1 h^3(p) - c_2 h^2(p) - c_3.
$$

Thus h is bounded, hence $|\nabla^2 w|$ is bounded. $\square$

We now return to the a priori estimate of solution of equation $(*)_\delta$. The main point is to modify the proof of Theorem 8.10 by applying an integral form of the Pogorelov estimate.

Proposition 8.14 *There is $\delta_0 \geq 0$, and $C = C(g)$, such that for all $\delta \leq \delta_0, w \in C^\infty(M)$ solving $(*)_\delta$ with $R_{g_w} > 0$ and $\int_M \sigma(A_{g_w})\, dg_w > 0$, the following estimate holds,*

$$
\int_M |\nabla_0^2 w|_0^3\, dv_0 + \int_M |\nabla_0 w|_0^{12}\, dv_0 \leq C. \tag{8.26}
$$

In particular, there is $\alpha > 0$, such that

$$
||w||_{C^\alpha} \leq C(g).
$$

The crucial step of the proof is in the following lemma:

Lemma 8.15 (Main Lemma) *There are constants $\delta_0 \geq 0, C = C(g_0)$, such that in terms of $g_w = e^{2w} g_0$,*

$$\begin{aligned} \frac{\delta}{16} \int_M \frac{(\Delta R)^2}{R}\, dv + \int_M \left(\frac{R}{6}\right)^3 dv \\ \leq (1 + c\delta) \int_M |\nabla w|^6\, dv + c \int_M R^2\, dv + c. \end{aligned} \tag{8.27}$$

Instead of the pointwise maximum principle as in the proof of Theorem 8.10 we use integral estimates. Denote

$$I = \int_M S^{ij} \nabla_i \nabla_j R\, dv,$$

$$II\colon = \int_M S^{ij} \nabla_i \nabla_j V\, dv$$

for $V\colon = \frac{1}{2}|\nabla w|^2$, where here and in the following, $dv = dv_{g_w}$ and all covariant derivatives are taken with respect to the metric g_w unless otherwise noted.

We remark that due to the fact that $\nabla_i S_{ij} = 0$, we have both $I = II \equiv 0$.

We also remark that in contrast to the proof of Theorem 8.10 we now only have $|\nabla w| \in L^4(M)$ and $w \geq c$ for w satisfies $(*)_\delta$.

Lemma 8.16 *There is a constant $C = C(g_0)$, such that*

$$\mathrm{I} \geq \int_M \left(\frac{3}{2}\delta \frac{(\Delta R)^2}{R} + 6TrE^3 + \frac{1}{12}R^3 - CR^2 - C \right) dv, \tag{8.28}$$

for any $w \in C^\infty(M)$ solving $(*)_\delta$.

Lemma 8.17 *There is a constant $C = C(g_0)$, such that*

$$\begin{aligned} \mathrm{II} \geq \int_M \Big(-\frac{1}{4}TrE^3 + \frac{1}{288}R^3 - \frac{1}{4}R|\nabla w|^4 \\ - C\delta R^3 - C\delta |\nabla w|^6 - CR^2 - C \Big)\, dv \end{aligned} \tag{8.29}$$

for all $w \in C^\infty(M)$ solving $(*)_\delta$.

Assuming (8.28), (8.29) for a moment, we will finish the proof of (8.27) in Lemma 8.15. In fact

$$\begin{aligned} 0 = \mathrm{I} + 24\mathrm{II} \geq \frac{3}{2}\delta \int_M \frac{(\Delta R)^2}{R}\, dv + \frac{1}{6}\int_M R^3\, dv \\ - 6\int_M R|\nabla w|^4\, dv - \int_M (C\delta R^3 + C\delta|\nabla w|^6 + CR^2 + C)\, dv. \end{aligned}$$

Divide by 36 and apply Hölder's and Young's inequality to get

$$\begin{aligned}
\frac{\delta}{24}\int_M \frac{(\Delta R)^2}{R}\,dv + \int_M \left(\frac{R}{6}\right)^3 dv &\le \int_M \left(\frac{R}{6}\right)|\nabla w|^4\,dv \\
&\quad + \frac{C\delta}{36}\int_M R^3\,dv + \frac{C\delta}{36}\int_M |\nabla w|^6\,dv + \frac{C}{36}\int_M (R^2+1)\,dv \\
&\le \left(\int_M \left(\frac{R}{6}\right)^3 dv\right)^{\frac{1}{3}} \left(\int_M |\nabla w|^6\,dv\right)^{\frac{2}{3}} + \cdots \\
&\le \frac{1}{3}\int_M \left(\frac{R}{6}\right)^3 dv + \frac{2}{3}\int_M |\nabla w|^6\,dv + \cdots,
\end{aligned}$$

where the dots denote the remaining terms on the right-hand side. Absorbing the first term on the right into the left-hand side finishes the proof of Lemma 8.15. □

Proof of (8.28): Integrate (8.14) in Lemma 8.11 and use (8.18), (8.19) to get (in terms of the metric g_w)

$$\begin{aligned}
\mathrm{I} &= 3\int_M \Bigg(\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) + 6TrE^3 \\
&\quad + R|E|^2 - 6WEE - 6BE\Bigg)\,dv \\
&\ge 3\int_M \left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) dv + \int_M 6TrE^3\,dv \\
&\quad + \int_M (CR^2 + C)\,dv + \int_M R|E|^2\,dv,
\end{aligned} \tag{8.30}$$

where we have used that

$$0 < \int_M \sigma_2\,dv = \frac{1}{2}\int_M \left(\frac{R^2}{12} - |E|^2\right) dv, \quad \text{whence} \quad \int_M |E|^2\,dv \lesssim \int_M R^2\,dv.$$

To estimate $\int_M R|E|^2\,dv$ from below, recall $(*)_\delta$,

$$\delta\Delta R = 4\sigma_2 + 8\gamma_1|W|^2,$$

where $\gamma_1 < 0$, since $\int_M \sigma_2\,dv > 0$; compare to Chapter 6.

Multiplication of $(*)_\delta$ by R and integration leads to

$$\delta\int_M R\Delta R\,dv = \int_M \frac{1}{6}R^3\,dv - 2\int_M R|E|^2\,dv + 8\gamma_1\int_M R|W|^2\,dv, \text{ i.e.,}$$

$$\begin{aligned}
\int_M R|E|^2\,dv &= \frac{1}{12}\int_M R^3\,dv + 4\gamma_1\int_M R|W|^2\,dv + \frac{\delta}{2}\int_M |\nabla R|^2\,dv \\
&\ge \frac{1}{12}\int_M R^3\,dv - C\int_M (R^2+1)\,dv.
\end{aligned} \tag{8.31}$$

Finally, to handle the first term on the right of (8.30) we claim that

$$\int_M \left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) dv \geq \frac{1}{2}\int_M \delta \frac{(\Delta R)^2}{R}\, dv - C, \tag{8.32}$$

which together with (8.31) inserted into (8.30) proves (8.28).

To prove (8.32) we differentiate $(*)_\delta$ and get

$$\delta \nabla \Delta R = \frac{1}{3} R \nabla R - 4|E|\nabla(|E|) - 8\gamma_1 \nabla(|W|^2),$$

multiply this by $\frac{\nabla R}{R}$ and integrate. □

The proof of (8.29) is a modification of (8.21) in Lemma 8.13, and we will skip the details here [23].

We will now apply Lemma 8.15 to prove Proposition 8.14.

Sketch of the proof of Proposition 8.14. Basically we are going to apply interpolation and boot-strapping methods to estimate the norms w. To do so, we first recall (7.3)

$$R = e^{-2w} R_0 - 6\Delta w + 6|\nabla w|^2.$$

Also

$$\begin{aligned} |\nabla w| &= |\nabla_0 w| e^{-w}, \text{or } |\nabla_0 w| = |\nabla w| e^{w}, \\ |\nabla_0^2 w|^2 &\lesssim |\nabla^2 w|^2 e^{4w} + e^{4w}|\nabla w|^4, \\ dv_0 &= e^{-4w}\, dv, \\ \left(\int_M |f|^{12}\, dv_0\right)^{\frac{1}{4}} &\lesssim \int_M |\nabla_0 f|_0^3\, dv_0 + \int_M |f|^3\, dv_0, \end{aligned} \tag{8.33}$$

the latter resulting from the Sobolev embedding $W^{1,3}(M) \hookrightarrow L^{12}(M)$.

Step a. We claim that

$$\left(\int_M |\nabla w|^{12}\, dv\right)^{\frac{1}{4}} \lesssim \int_M |\nabla w|^6\, dv + 1. \tag{8.34}$$

Proof. Taking $f := |\nabla_0 w| e^{-\frac{2}{3}w}$ in (8.33) one gets

$$\int_M |f|^{12}\, dv_0 = \int_M |\nabla_0 w|^{12} e^{-8w}\, dv_0 = \int_M |\nabla w|^{12}\, dv,$$

whence by (8.33)

$$\begin{aligned} \left(\int_M |\nabla w|^{12}\, dv\right)^{\frac{1}{4}} &\lesssim \int_M |\nabla_0(|\nabla_0 w| e^{-\frac{2}{3}w})|^3\, dv_0 + \int_M |\nabla_0 w|^3 e^{-2w}\, dv_0 \\ &\lesssim \int_M \left(|\nabla_0^2 w|^3 e^{-2w} + |\nabla_0 w|^6 e^{-2w}\right) dv_0 + C \\ &\lesssim \int_M |\nabla^2 w|^3\, dv + \int_M |\nabla w|^6\, dv + 1. \end{aligned}$$

Now, by (7.6) and (7.1)

$$|\nabla^2 w|^3 \lesssim |A|^3 + |\nabla w|^6 + C,$$
$$|A|^2 = |E|^2 + \frac{R^2}{36}.$$

Thus

$$\begin{aligned}
\left(\int_M |\nabla w|^{12}\,dv\right)^{\frac{1}{4}} &\lesssim \int_M (|A|^3 + |\nabla w|^6 + 1)\,dv \\
&\lesssim \int_M (|E|^3 + R^3 + |\nabla w|^6 + 1)\,dv \\
&\underset{(*)_\delta}{\lesssim} \int_M (\delta|\nabla E|^2 + \delta|\nabla R|^2 + R^3 + |\nabla w|^6 + 1)\,dv \qquad (8.35)\\
&\lesssim \delta\int_M |\nabla R|^2\,dv + \int_M (R^3 + |\nabla w|^6 + 1)\,dv \\
&\underset{(8.27)}{\lesssim} \int_M (|\nabla w|^6 + 1)\,dv.
\end{aligned}$$

Notice that we used $(*)_\delta$ to express $|E|^3$ in terms of $|\nabla R|^2$. To be more precise, multiplying $(*)_\delta$ by E and integrating one gets

$$\int_M |E|^3\,dv \lesssim \left(\int_M R^3\,dv\right)^{\frac{2}{3}}\left(\int_M E^3\,dv\right)^{\frac{1}{3}} + \varepsilon\int_M E^3\,dv + \frac{C}{\varepsilon} + \frac{\delta}{2}\int_M |\nabla R||\nabla E|\,dv,$$

for some small $\varepsilon > 0$, hence

$$\int_M |E|^3\,dv \lesssim \int_M R^3\,dv + \int_M |\nabla E|^2\,dv + \int_M |\nabla R|^2\,dv + C.$$

Note also that we used

$$\begin{aligned}
\delta\int_M |\nabla R|^2\,dv &= \delta\int_M (-\Delta R)R\,dv \\
&\le \delta\int_M \frac{(\Delta R)^2}{R}\,dv + \delta\int_M R^3\,dv \\
&\underset{(8.27)}{\lesssim} \int_M (|\nabla w|^6 + 1)\,dv
\end{aligned}$$

in the last step of (8.35). □

Step b. Claim:

$$\int_M |\nabla^2 w|^2|\nabla w|^2\,dv \lesssim \int_M (\delta|\nabla w|^6 + R^2 + 1)\,dv. \qquad (8.36)$$

Proof. Recall (7.3) which implies

$$\frac{R}{6} = -\Delta w + |\nabla w|^2 + \frac{1}{6} R_0 e^{-2w}. \tag{8.37}$$

The key observation is

$$\int_M |\nabla w|^6 \, dv \leq \frac{1}{6} \int_M R |\nabla w|^4 \, dv + C \int_M (\delta R^3 + \delta |\nabla w|^6 + R^2 + 1) \, dv. \tag{8.38}$$

Assuming (8.38) for the moment we can conclude

$$\int_M \Delta w |\nabla w|^4 \, dv \leq C\delta \int_M R^3 \, dv + C \int_M (\delta |\nabla w|^6 + R^2 + 1) \, dv, \tag{8.39}$$

thus (by multiplication of the square of (8.37) with $|\nabla w|^2$),

$$\begin{aligned} \int_M (\Delta w)^2 |\nabla w|^2 &\leq \int_M \left(\left(\frac{R}{6}\right)^2 |\nabla w|^2 - |\nabla w|^6 + 2\Delta w |\nabla w|^4 \right) dv \\ &\quad + C \int_M (R^2 + 1) \, dv \\ &\underset{(8.39)}{\leq} \delta \int_M |\nabla w|^6 \, dv + C \int_M (R^2 + 1) \, dv. \end{aligned}$$

By Bochner's formula we finally obtain

$$\int_M |\nabla^2 w|^2 (\nabla w)^2 \lesssim \delta \int_M |\nabla w|^6 \, dv + C \int_M (R^2 + 1) \, dv.$$

To see (8.38) recall from Lemma 7.2 (c) that $\mathrm{Ric} \geq \frac{3\sigma_2}{R}$, so that

$$\begin{aligned} 2 \int_M |\nabla w|^2 \operatorname{Ric}(\nabla w, \nabla w) \, dv &\geq \int_M \frac{6\sigma_2}{R} |\nabla w|^4 \, dv \\ &\underset{(*)_\delta}{\geq} -6\delta \int_M |\nabla^2 w|^2 |\nabla w|^2 \, dv \\ &\gtrsim -\delta \int_M R^3 \, dv - \delta \int_M |\nabla w|^6 \, dv - \int_M (R^2 + 1) \, dv. \end{aligned}$$

On the other hand,

$$\begin{aligned} 2 \int_M |\nabla w|^2 \operatorname{Ric}(\nabla w, \nabla w) \, dv &= \frac{1}{6} \int_M (R |\nabla w|^4 - |\nabla w|^6) \, dv \\ &\quad + \frac{1}{6} \int_M R_0 e^{-2w} |\nabla w|^4 \, dv \\ &\quad + 2 \int_M |\nabla w|^2 A_0(\nabla w, \nabla w) \, dv, \end{aligned}$$

where the last two terms are bounded by virtue of (8.2) in Proposition 8.5. □

Step c. To estimate $\int_M |\nabla w|^6\, dv$ we proceed as follows:

$$\begin{aligned}
\int_M |\nabla w|^6\, dv &= \int_M \langle \nabla w, \nabla w\rangle |\nabla w|^4\, dv \\
&= -\int_M w\Delta w|\nabla w|^4\, dv - \int_M w\nabla w\nabla(|\nabla w|^4)\, dv \\
&\lesssim \int_M |w||\nabla^2 w||\nabla w|^4\, dv \\
&\lesssim \left(\int_M |\nabla^2 w|^2|\nabla w|^2\, dv\right)^{\frac12}\left(\int_M |\nabla w|^6 w^2\, dv\right)^{\frac12} \\
&\lesssim \left(\int_M |\nabla^2 w|^2|\nabla w|^2\, dv\right)^{\frac12}\left(\int_M |\nabla w|^{12}\, dv\right)^{\frac18}\left(\int_M |\nabla w|^4|w|^{\frac83}\, dv\right)^{\frac38} \\
&\underset{(8.4),(8.34)}{\lesssim} \left(\int_M |\nabla^2 w|^2|\nabla w|^2\, dv\right)^{\frac12}\left(1+\int_M |\nabla w|^6\, dv\right)^{\frac12}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_M |\nabla w|^6\, dv &\lesssim \int_M |\nabla^2 w|^2|\nabla w|^2\, dv + 1 \\
&\underset{(8.36)}{\lesssim} \delta\int_M |\nabla w|^6\, dv + \int_M (R^2+1)\, dv,
\end{aligned}$$

which implies

$$\begin{aligned}
\int_M |\nabla w|^6\, dv &\lesssim \int_M R^2\, dv + 1 \\
&\lesssim \left(\int_M R^3\, dv\right)^{\frac23} + 1 \\
&\underset{(8.27)}{\lesssim} \left(\int_M |\nabla w|^6\, dv\right)^{\frac23} + 1,
\end{aligned}$$

i.e., $\int_M |\nabla w|^6\, dv \le C$, and by (8.34), $\int_M |\nabla w|^{12}\, dv \lesssim C$, and $\int_M |\nabla^2 w|^3\, dv \lesssim C$. □

Corollary 8.18 *There is a constant $C = C(g_0)$, such that*

$$\delta\int_M \frac{(\Delta R)^2}{R^2}\, dv \le C. \tag{8.40}$$

Proof. We know already that

$$\delta\int_M \frac{(\Delta R)^2}{R}\, dv \lesssim \int_M R^3\, dv + 1 \lesssim C.$$

Thus it suffices to show $\min_M R(\cdot) \geq c_0 > 0$, which will follow from the maximum principle applied to

$$\delta \Delta R = 8\gamma_1 |W|^2 + \frac{1}{6}R^2 - 2|E|^2 \leq 8\gamma_1 |W|^2 + \frac{1}{6}R^2.$$

Hence at the minimum point $p \in M$ of R we have $\Delta R(p) \geq 0$ and therefore

$$\frac{1}{6}R^2(p) \geq -8\gamma_1 |W|^2(p) \geq 8|\gamma_1| \min_M |W|^2(\cdot).$$

So if $|W|^2 \underset{(5.9)}{=} e^{-4w}|W|_0^2 \neq 0$ on M, then we are done, since then

$$R^2 \geq 48|\gamma_1| \min_M |W|^2(\cdot) =: c_0.$$

If $|W| = 0$ somewhere, choose a section $\eta \in \Gamma(\mathrm{Sym}(T^*M^4 \otimes T^*M^4))$, which denotes the bundle of symmetric $(0,2)$-tensors on M^4, e.g., $\eta =$ any Riemannian metric on M^4. Then $|\eta|^2 = e^{-4w}|\eta|^2_{g_0}$, and we look at the equation

$$\delta \Delta R = 4\sigma_2 + 8\gamma_1 |\eta|^2, \qquad (**)_\delta$$

and apply the maximum principle as above.

Notice that the only relevant fact about $|W|^2$ we used was the behavior under conformal change, see (5.9). So instead of $\mathrm{I}[w]$ in the definition of $F[w]$ or $F_\delta[w]$ one uses

$$\mathrm{I}'[w]\colon = 4\int_M w|\eta|^2\, dv - \int_M |\eta|^2\, dv \log ⨍_M e^{4w}\, dv. \qquad \square$$

We conclude with

Proposition 8.19 *There is a constant $\delta_0 < 1$ such that for each $s \in [0,5)$ there is a constant $C = C(s, .g_0)$, such that for all $0 < \delta \leq \delta_0$ the following holds:*

Any solution $w_\delta \in C^\infty(M)$ of $(**)_\delta$ with $R_{g_w} > 0$, $\int_M w\, dv_0 = 0$, $\int_M \sigma_w(A_{g_w})\, dv_{g_w} > 0$ satisfies

$$\int_M |\nabla_0^2 w|^s\, dv_0 \leq C.$$

We will skip the details of the proof here. [23] The idea of the proof is to apply the same arguments as above to the terms

$$\mathrm{I}\colon = \int_M S^{ij}\nabla_i \nabla_j R^{p+1}\, dv = 0 \qquad \text{and} \qquad \mathrm{II}\colon = \int_M S^{ij}\nabla_i (R^p \nabla_j V)\, dv = 0,$$

for $p < 2$.

As an immediate consequence we deduce from Sobolev's embedding theorem

Corollary 8.20 *There is a constant $\delta_0 < 1$, such that for each $\alpha \in (0,1)$ there is a constant C_α, such that the following holds: for all $\delta \in (0, \delta_0]$, any solution $w_\delta \in C^\infty(M)$ of $(**)_\delta$ with $R_{g_w} > 0, \int_M w\, dv_0 = 0, \int_M \sigma_2(A_{g_w})\, dv_{g_w} > 0$ satisfies*

$$||w||_{C^{1,\alpha}} \leq C_\alpha.$$

§ 9 Smoothing via the Yamabe flow

Theorem 9.1 *Let $g = e^{2w} g_0$ be a solution of $(**)_\delta$ with positive scalar curvature, normalized so that $\int w dv_0 = 0$. Assume also $\int \sigma_2(A_0) dv_0 > 0$. Then for δ sufficiently small, there exists $v \in C^\infty(M)$, such that $\sigma_2(A_h) > 0$ for $h = e^{2v} g$.*

The key step is to look at the evolution of the quantity k/R under the Yamabe flow, where

$$k \colon = \sigma_2 + 2\gamma_1 |\eta|^2, \tag{9.1}$$

$|\eta| > 0$, on M, and $|\eta|_{g_w} = e^{-2w}|\eta|$. Notice that by $(**)_\delta$, $\delta \Delta R = 4k$. We will assume an a priori bound in $L^p, p > 4$, for the curvature of the initial data. Throughout Chapter 9 we assume that the hypotheses of Theorem 9.1 hold.

Proposition 9.2 *Consider*

$$\begin{cases} \frac{\partial h}{\partial t} & = -\frac{1}{3} R h, \\ h(0,\cdot) & = g \colon = e^{2w} g_0. \end{cases} \tag{9.2}$$

Then there exists $T_0 = T_0(g_0)$, such that (9.2) *has a unique smooth solution $h \in C^\infty([0, T_0), M)$.*

Proof. Consider the *normalized Yamabe flow*

$$\begin{cases} \frac{\partial h^*}{\partial t} & = -\frac{1}{n-1}(R - r) h^*, \\ r(t) & = \int_M R\, dv / \int_M \, dv, \\ h^*(0,\cdot) & = h_0^*, \end{cases} \tag{9.3}$$

on (M^n, h_0). Then (9.3) admits a unique smooth solution for all time (see [58], [94]). When $n = 4$, (9.2) and (9.3) differ only by a rescaling in time and space. (9.3) guarantees that the volume is normalized, hence we are only required to find a time interval $[0, T_0(g_0))$, on which $\mathrm{vol}(M, h)$ is under control.

Some basic facts about the Yamabe flow are summarized in

Lemma 9.3 ([94]) *Under* (9.2) *one has*

$$\frac{\partial}{\partial t}(dv) = -\frac{2}{3} R\, dv, \tag{9.4}$$

$$\frac{\partial}{\partial t} R = \Delta R + \frac{1}{3} R^2, \tag{9.5}$$

$$\frac{\partial}{\partial t} R_{ij} = \frac{1}{3} \nabla_i \nabla_j R + \frac{1}{6} (\Delta R) g_{ij}. \tag{9.6}$$

Assuming the validity of (9.4)–(9.6), we now finish the proof of Proposition 9.2 as follows:

Since by $(**)_\delta$ $R_g = R_{h(0,\cdot)} > C(g_0) > 0$, we infer from (9.5) that at a minimum point $p_t \in M$,

$$\frac{\partial R}{\partial t}(p_t) = \Delta R(p_t) + \frac{1}{3}R^2(p_t) \geq \frac{1}{3}R^2(p_t) > 0,$$

hence R remains positive under the flow.

The volume is decreasing, since by (9.4)

$$\frac{d}{dt}\int_M dv = -\frac{2}{3}\int_M R\, dv < 0.$$

In addition,

$$\frac{d}{dt}\int_M dv \geq -\frac{2}{3}\left(\int_M R^2\, dv\right)^{\frac{1}{2}}\left(\int_M dv\right)^{\frac{1}{2}},$$

whence

$$\frac{d}{dt}\left(\int_M dv\right)^{\frac{1}{2}} \geq -\frac{1}{3}\left(\int_M R^2\, dv\right)^{\frac{1}{2}}. \tag{9.7}$$

On the other hand, by (9.4) and (9.5),

$$\begin{aligned}\frac{d}{dt}\int_M R^2\, dv &= \int_M 2R\frac{dR}{dt}\, dv + \int_M R^2 \frac{d}{dt}(dv)\\ &= \int_M 2R\left(\Delta R + \frac{1}{3}R^2\right)\, dv + \int_M R^2\left(-\frac{2}{3}R\right)\, dv \qquad (9.8)\\ &= -2\int_M |\nabla R|^2\, dv \leq 0.\end{aligned}$$

(9.7) and (9.8) imply

$$\left[\mathrm{vol}(M, h(0,\cdot))^{\frac{1}{2}} - \frac{||R_g||_{L^2}}{3}\cdot t\right]^2 \leq \mathrm{vol}(M, h(t,\cdot)) \leq \mathrm{vol}(M, h(0,\cdot)),$$

and $||R_g||_{L^2}$ is bounded according to Proposition 8.14.

Proposition 9.4 *Fix $s \in (4,5)$. Then there is $T_1 = T_1(g_0) < T_0$, such that for $t \leq T_1$ the solution $h = e^{2v}g$ of (9.2) satisfies*

(a) $||\operatorname{Ric}_h||_{L^s} \leq 2||\operatorname{Ric}_g||_{L^s}$,

(b) $||\operatorname{Ric}_h||_{L^\infty} \leq C_2 t^{-\frac{2}{s}}$, *where* $C_2 = C_2(g_0)$,

(c) $||v||_{L^\infty} \leq C(g_0)$.

Proof. The proof relies on general estimates for the Yamabe flow (see [93]) as a parabolic evolution equation summarized in

Proposition 9.5 (Moser iteration for parabolic equations, see [93]). *Assume that with respect to the metric $h(t), 0 \leq t \leq T$ the following Sobolev inequality holds:*

$$\left(\int_M |\varphi|^{\frac{2n}{n-2}}\, dv\right)^{\frac{n-2}{n}} \leq C_S\left[\int_M |\nabla\varphi|^2\, dv + \int_M \varphi^2\, dv\right]$$

for all $\varphi \in W^{1,2}(M^n)$. Suppose b is a nonnegative function on $[0,T] \times M^n$, such that

$$\frac{\partial}{\partial t}(dv) \leq b\, dv.$$

Let $q > n$, and $u \geq 0$ be a function satisfying

$$\frac{\partial u}{\partial t} \leq \Delta u + bu,$$
$$\sup_{0 \leq t \leq T} ||b||_{L^{q/2}} \leq \beta.$$

Then for all $p_0 > 1$, there exists a constant $C = C(n, q, p_0, C_S)$ such that for $0 \leq t \leq T$,

$$||u(t,\cdot)||_{L^\infty} \leq Ce^{Ct}t^{-\frac{n}{2p_0}}||u(0,\cdot)||_{L^{p_0}}.$$

Moreover, for given $p \geq p_0 > 1$, one has for all $t \in [0,T]$,

$$\frac{d}{dt}\int_M u^p\, dv + \int_M |\nabla(u^{p/2})|^2\, dv \leq Cp^{\frac{2n}{q-n}}\int_M u^p\, dv,$$

where $C = C(n, q, p_0, C_S)$.

Remark 9.6 When applying Proposition 9.5 to prove Proposition 9.4, we only require that $s > \frac{n}{2} = 2$ for $n = 4$. Also, in our application, we can control the Sobolev constant C_S by the Yamabe constant $Y(M, g_0)$ which we assume to be positive of (M, g_0)[23]. □

The following result contains the key inequality for the proof of Theorem 9.1.

Proposition 9.7 *For k as defined in* (9.1)*, denote*

$$\varphi := \max\left(-\frac{k}{R}, 0\right).$$

Then for $t \leq T_1$

$$\frac{\partial\varphi}{\partial t} \leq \Delta\varphi + C_1|\mathrm{Ric}|\varphi + C_1|\mathrm{Ric}| \tag{9.9}$$

for some constant $C_1 = C_1(g_0)$.

Proof. This statement is proved by straightforward but lengthy computations, we refer to [23]. □

Now we are going to sketch the proof of Theorem 9.1.

First we will modify φ to " remove " the last term in (9.9). For this purpose, we define $\varphi_1(t) := \exp\left(\frac{s}{s-2} C_1 C_2 t^{\frac{s-2}{s}}\right) - 1$, hence $\varphi_1(0) = 0$, and since $s > 2$, one easily checks that

$$\frac{\partial \varphi_1}{\partial t}(t) = C_1 C_2 (1 + \varphi_1(t)) t^{-\frac{2}{s}}.$$

Then $u := \varphi - \varphi_1$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi_1}{\partial t} \\ &\underset{(9.9)}{\leq} \Delta\varphi + C_1|\mathrm{Ric}|\varphi + C_1|\mathrm{Ric}| - \frac{\partial \varphi_1}{\partial t} \\ &= \Delta u + C_1|\mathrm{Ric}|u + C_1|\mathrm{Ric}|\varphi_1 + C_1|\mathrm{Ric}| - \frac{\partial \varphi_1}{\partial t} \\ &\underset{(\text{Prop. } (9.4)(\text{b}))}{\leq} \Delta u + C_1|\mathrm{Ric}|u + C_1 C_2 (1+\varphi_1) t^{-\frac{2}{s}} - \frac{\partial \varphi_1}{\partial t} \\ &= \Delta u + C_1|\mathrm{Ric}|u. \end{aligned}$$

Applying Proposition 9.5 for $b = c_1|\mathrm{Ric}|, p_0 = 2, q = 2s, s > 4$, we conclude for $t \leq T_1$,

$$\begin{aligned} ||u||_{L^\infty} = ||\varphi - \varphi_1||_{L^\infty} &\leq C t^{-1} ||\varphi(0,\cdot) - \varphi_1(0)||_{L^2} \\ &= \frac{C}{t} ||\varphi(0,\cdot)||_{L^2}. \end{aligned}$$

On the other hand, by $(**)_\delta$,

$$\begin{aligned} ||\varphi(0,\cdot)||_{L^2} &= \left|\left| \frac{\sigma_2(A) + 2\gamma_1|\eta|^2}{R} \right|\right|_{L^2} \\ &\underset{(**)_\delta}{=} \left|\left| \frac{\delta}{4} \frac{\Delta_g R_g}{R_g} \right|\right|_{L^2} \\ &\underset{(8.40)}{\leq} C(g_0)\delta^{\frac{1}{2}}. \end{aligned}$$

Thus $||u||_{L^\infty} = ||\varphi - \varphi_1||_{L^\infty} \leq \frac{C\delta^{\frac{1}{2}}}{t}$ for all $t \leq T_1$. That is, by definition of φ in Proposition 9.7,

$$\frac{1}{R}(\sigma_2 + 2\gamma_1|\eta|^2) \geq -\varphi_1(t) - \frac{C\delta^{\frac{1}{2}}}{t},$$

hence

$$\sigma_2 + 2\gamma_1|\eta|^2 \geq R\left(-\varphi_1(t) - C\frac{\delta^{\frac{1}{2}}}{t}\right) \geq C t^{-\frac{2}{s}} \left(-t^{1-\frac{2}{s}} - \delta^{\frac{1}{2}} t^{-1}\right),$$

since $R \leq Ct^{-\frac{2}{s}}$ by Proposition 9.4 (b), and $\varphi_1(t) \leq Ct^{1-\frac{2}{s}}$ by the simple estimate $e^x - 1 \leq |x|e^{|x|}$ for $t \leq T_1$.

Consequently,

$$\sigma_2 \geq -2\gamma_1|\eta|^2 - C_3 t^{1-\frac{4}{s}} - C_3\delta^{\frac{1}{2}} t^{-1-\frac{2}{s}}.$$

Recall that $|\eta|^2 = e^{-4(v+w)}|\eta|_0^2 \geq C(g_0) > 0$, by Proposition 9.4 (c). Hence there is a constant $C_4 = C_4(g_0) > 0$ so that $\sigma_2(A_t) \geq C_4 - C_3 t^{1-\frac{4}{s}} - C_3\delta^{\frac{1}{2}} t^{-1-\frac{2}{s}}$ for all $t \leq T_1$.

Let $t_0 \colon = \min\{T_1, \hat{t}_0\}$, where $\hat{t}_0$ is chosen such that

$$C_3 \hat{t}_0^{(1-\frac{4}{s})} = \frac{1}{4} C_4,$$

then at $t = t_0$,

$$\sigma_2(A_{t_0}) \geq \frac{3}{4} C_4 - C_3\delta^{\frac{1}{2}} t_0^{-1-\frac{2}{s}} > \frac{1}{2} C_4,$$

if $\delta < \delta_0$ is sufficiently small. This means that the metric $h = h(t_0, \cdot) \in C^\infty(M)$ satisfies

$$\sigma_2(A_{t_0}) = \sigma_2(A_{h(t_0,\cdot)}) > 0. \qquad \square$$

§ 10 Deforming σ_2 to a constant function

In this section we will outline the result in [24]. The goal is to deform $\sigma_2 = f$, where $f \in C^\infty(M), f > 0$, into $\sigma_2 = c$, where $c > 0$ is a constant on a compact 4-manifold. To achieve this, we will use the method of continuity together with a degree-theoretic argument.

To apply the method of continuity, the main step is to obtain a priori estimates for solutions w of the equation $\sigma_2(A_{g_w}) = f$ for a given positive function f. First we observe that on (S^4, g_c), due to the noncompactness of the diffeomorphism group on S^4, we do not have an a priori sup-norm bound of the conformal factor for w with $\sigma_2(A_{g_w}) \equiv 6$. That is, if we consider the family of metrics $g_w = e^{2w} g_c$ on S^4 defined by $e^{2w} g_c = \phi^* g_c$ for some diffeomorphism ϕ of S^4 (actually we can take ϕ to be a rotation and dilation on S^4), then $R_{g_w} \equiv 12$, $E_{g_w} \equiv 0$ and

$$\sigma_2(A_{g_w}) = \frac{1}{2}\frac{1}{12}(4 \cdot 3)^2 = 6 \quad \text{on } S^4.$$

To see that there is no a priori sup-norm bound of such a family of w, we may use the stereographic projection map $S^4 - \{\mathbb{N}\}$ to $\mathbb{R}^4$, where $\mathbb{N}$ is the north pole and observe that in Euclidean coordinates on $\mathbb{R}^4$, w corresponds to the sequence

$$w = w_\lambda = \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2}$$

with $\lambda > 0, x_0 \in \mathbb{R}^4$. Thus the supremum norm of w_λ tends to infinity as $\lambda \to 0$.

The following theorem indicates that (S^4, g_c) is the only exceptional case among all compact 4-manifolds.

Theorem 10.1 *On (M^4, g_0), suppose that $R_{g_w} > 0, g_w = e^{2w} g_0$, and*

$$\sigma_2(A_{g_w}) = f > 0$$

for some smooth function f. If (M^4, g_0) is not conformally equivalent to (S^4, g_c), then there is a constant $C = C(||f||_{C^3}, g_0, (\min f)^{-1})$, such that

$$\max_{M^4}(e^{w(\cdot)} + |\nabla_0 w|(\cdot)) \leq C. \tag{10.1}$$

Once the estimate (10.1) is established, we can apply Theorem 8.10 to establish $w \in C^{1,1}(M)$, and then since $(\sigma_2)^{\frac{1}{2}}$ is concave, we can apply the results of Evans [42] and Krylov [60] to establish that $w \in C^{2,\alpha}(M)$, hence $w \in C^\infty(M)$. That is, we have the following corollary.

Corollary A. *There is a constant C, such that $||w||_{C^\infty} \leq C$, if $f \in C^\infty(M)$.*

We then apply a degree-theoretic argument to deform σ_2 to a constant. We will skip this part of the argument in this note and refer the readers to the article [24].

Theorem 10.2 *Assume that* $\sigma_2(A_g) = f > 0$, *then there is a metric* $g_w = e^{2w}g$ *such that*

$$\sigma_2(A_{g_w}) \equiv 1.$$

Outline of the proof of Theorem 10.1

We will proceed in five steps:

Step 1. Given a sequence of functions $w_i \in C^\infty(M)$, such that (10.1) fails to hold we use a blow-up argument to construct a new sequence converging to a solution of $\sigma_2 \equiv 1$ or $\sigma_2 \equiv 0$ on $(\mathbb{R}^4, |dx|^2)$. The main technical difficulty is the absence of a Harnack inequality for solutions of $\sigma_2 = f > 0$. [5] Hence even if the suitably dilated sequence may be shown to be bounded from above, there is a lack of a lower bound.

Step 2. Classify the solutions of $\sigma_2 \equiv 0$ on $\mathbb{R}^4$ according to

Theorem 10.3 *Suppose* $g_w = e^{2w}|dx|^2$ *is a conformal metric on* $\mathbb{R}^4$ *with* $w \in C^{1,1}(\mathbb{R}^4)$ *satisfying*

$$\sigma_2(A_{g_w}) \equiv 0, R_{g_w} \geq 0;$$

then $w \equiv$ const.

Step 3. Classify the solutions of $\sigma_2 \equiv$ *constant* > 0 on $\mathbb{R}^4$ according to

Theorem 10.4 *Suppose* $g_w = e^{2w}|dx|^2 =: u^2|dx|^2$ *is a conformal metric on* $\mathbb{R}^4$ *with*

$$\sigma_2(A_{g_w}) \equiv 6 \quad (\Rightarrow R_{g_w} \equiv \pm 12);$$

then $u(x) = (a|x|^2 + \sum_{i=1}^4 b_i x_i + c)^{-1}$ *for some constants* a, b, c. *In particular,* g_w *is the pull-back of the round metric* g_c *on* S^4 *to* $\mathbb{R}^4$.

Step 4. The previous two steps together with the following important lemma by Gursky will be used to establish Theorem 10.1.

Lemma 10.5 [54] *Let* (M^4, g) *with* $Y(M^4, g) > 0$. *Then* $\int_M \sigma_2(A_g) dv_g \leq 16\pi^2$ *and equality holds if and only if* (M^4, g) *is conformally equivalent to* (S^4, g_c).

We remark that this is a restatement of Lemma 6.12 in Section 6. As on (M^4, g) we have

$$Q_g = -\frac{1}{12}\Delta R_g + \frac{1}{2}\sigma_2(A_g).$$

Hence

$$k_g := \int_M Q_g dv_g = \frac{1}{2}\int_M \sigma_2(A_g) dv_g.$$

Thus $\int_M \sigma_2(A_g) dv_g \leq 16\pi^2$ if and only if $k_g \leq 8\pi^2$.

[5] After this note was written, a form of Harnack inequality was established for a class of fully nonlinear elliptic equations defined on R^n which includes the σ_k equations. The reader is referred to the recent articles of [52] and [62].

Remarks

1. Step 3 above works also for $\sigma_2(A_g) \equiv \text{const.}$ on $\mathbb{R}^n$ for $n = 4, 5$, and for $n \geq 6$ under the additional assumption that $\int_M dv_g < \infty$. For $n = 4, \sigma_2 > 0$ and $R > 0$ imply that $\int dv_g < \infty$. We remark that for $n \geq 5$ there is a metric with $\sigma_2 > 0, R > 0$ with $\int dv_g$ unbounded (obtained by a perturbation of a metric on $S^{n-1} \times S^1$), see the article [25].
2. The classification result of Step 3 should be compared to the result of Caffarelli–Gidas–Spruck [16] for

$$-\Delta u = c_n u^{\frac{n+2}{n-2}} \text{ on } \mathbb{R}^n$$
$$\Rightarrow u = \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}.$$

On (S^n, g_c) the above result is Obata's [71] theorem, which states that if $u > 0$ satisfies

$$-\Delta u + R_0 u = c u^{\frac{n+2}{n-2}} \quad \text{on } S^n$$

for $R_0 = n(n-1)$, then $u^{\frac{4}{n-2}} g_c = \phi^* g_c$ for a conformal transformation $\phi : S^n \to S^n$.

Such a classification result has been established by J. Viaclovsky [90] for general σ_k (see also Corollary 8.12 for $k = 2$ on S^4):

Theorem 10.6 (Viaclovsky [90]) *If $\sigma_k(A_g) \equiv \text{const.}$ on S^n for $g = u^{\frac{4}{n-2}} |dx|^2$, then $u = (a|x|^2 + b_i x_i + c)^{-\frac{2}{n-2}}$ for some constants a, b, c.*

Step 1. We will use an unusual blow-up sequence w_k, since we do not have a Harnack inequality to derive a lower bound on w_k once we have an upper bound.

Assuming that the statement (10.1) is not true, we find a sequence of metrics $g_k = e^{2w_k} g_0$, and smooth functions f_k, such that $\sigma_2(A_{g_k}) = f_k$ with $0 < C_0 \leq f_k \leq C_0^{-1}$ and $||f_k||_{C^2} \leq C_1$, such that

$$\max_M (e^{w_k} + |\nabla_0 w_k|) \to \infty \text{ as } k \to \infty. \tag{10.2}$$

Assume that $p_k \in M$ are the corresponding maximum points. Choosing normal coordinates Φ_k at p_k we may identify a neighborhood of p_k with the unit ball $B_1(0) \subset \mathbb{R}^4$ with $\Phi_k(p_k) = 0 \in \mathbb{R}^4$. Define dilations

$$T_\varepsilon \colon \mathbb{R}^4 \longrightarrow \mathbb{R}^4,$$
$$x \longrightarrow T_\varepsilon(x) \colon = \varepsilon x,$$

and consider $w_{k,\varepsilon} = T_\varepsilon^* w_k + \log \varepsilon$; hence

$$\nabla_0 w_{k,\varepsilon} + e^{w_{k,\varepsilon}} = \varepsilon(\nabla_0 w_k + e^{w_k}) \circ T_\varepsilon.$$

Now choose for each $k, \varepsilon = \varepsilon_k$ such that the right-hand side equals 1 at $x = 0$, i.e.,

$$\nabla_0(w_{k,\varepsilon_k}) + e^{w_{k,\varepsilon_k}}{}_{|x=0} = 1, \tag{10.3}$$

then w_{k,ε_k} is defined on $B_{\frac{1}{\varepsilon_k}}(0)$.

Notice that $0 \in \mathbb{R}^4$ corresponds to a maximal point $p_k \in M$ for each k, with value normalized to 1 by (10.3), i.e., with

$$\nabla_0(w_{k,\varepsilon_k}) + e^{w_{k,\varepsilon_k}} \leq 1 \text{ on } B_{\frac{1}{\varepsilon_k}}(0). \tag{10.4}$$

Since the ε_k are chosen, we change notation by setting $w_k \colon = w_{k,\varepsilon_k}$ from now on. Denote the pull-back $g_k^* \colon = e^{2w_k} T_{\varepsilon_k}^* g_0$, then $\sigma_2(A_{g_k^*}) = f_k \circ T_{\varepsilon_k}$ with

$$g_0^k = T_{\varepsilon_k}^* g_0 \to |dx|^2$$

in the $C^{2,\beta}$-topology.

Case 1 $$\lim_{k\to\infty} e^{w_k(0)} = 0,$$

i.e., $w_k(0) \to -\infty$, then the shifted functions $\bar{w}_k \colon = w_k - w_k(0)$ with the corresponding metrics $\bar{g}_k \colon = e^{2\bar{w}_k} g_0$, satisfy

$$\begin{cases} \bar{w}_k(0) = 0, \\ |d\bar{w}_k| \leq 1 \text{ on } B_{\frac{1}{\varepsilon_k}}(0) \subset \mathbb{R}^4, \\ \lim_{k\to\infty} |d\bar{w}_k(0)| = 1, \\ \sigma_2(A_{\bar{g}_k^*}) = e^{4w_k(0)} f_k \circ T_{\varepsilon_k} \text{ on } B_{\frac{1}{\varepsilon_k}}(0) \subset \mathbb{R}^4. \end{cases} \tag{10.5}$$

Thus $\max_{B_\varrho(0)} |\bar{w}_k| \leq \varrho$, so the $\bar{w}_k$ are uniformly bounded in the C^1-topology on compact subsets of $\mathbb{R}^4$. To obtain the necessary $C^{1,1}$-bounds we appeal to a local version of Theorem 8.10 on $\mathbb{R}^4$:

Theorem 10.7 *Suppose $g = e^{2w}|dx|^2 =: e^{2w} g_0$ on $\mathbb{R}^4$ satisfies $\sigma_2(A_g) = f \geq 0$ and $R_g > 0$ on $B_\varrho(0)$; then*

$$|\nabla_0^2 w|_{L^\infty(B_{\varrho/2})} \leq C(||w||_{L^\infty(B_\varrho)}, ||\nabla_0 w||_{L^\infty(B_\varrho)}, ||f||_{C^2(B_\varrho)}, \varrho). \tag{10.6}$$

(10.6) implies in our situation

$$\sup_{B_\varrho(0)} |\nabla^2 \bar{w}_k| \leq C_\varrho. \tag{10.7}$$

Case 2 $$\limsup_{k\to\infty} e^{w_k(0)} = \delta_0 > 0;$$

then

$$\begin{cases} -c_2 & \leq w_k(0) \underset{(10.4)}{\leq} 0, \\ |dw_k| & \leq 1 \text{ on } B_{\frac{1}{\varepsilon_k}}(0). \end{cases} \tag{10.8}$$

Again as before we obtain

$$\sup_{B_\varrho(0)} |\nabla^2 w_k| \le C_\varrho .$$

In contrast to Case 1 we even get uniform $C^{2,\beta}$-bounds by the theory of Evans [42] and Krylov [60], since the w_k satisfy the uniformly elliptic equations

$$\sigma_2(A_{g_k}) = f_k \circ T_{\varepsilon_k} \ge \frac{1}{C_0}.$$

Recall that for the ellipticity one has to check that (by Lemma 7.2 (c))

$$-\frac{\partial \sigma_2(A_{g_k})}{\partial (w_k)_{ij}} = 2S_{ij} \ge \frac{6\sigma_2(A_{g_k})}{R_{g_k}} g_{ij}$$

which is uniformly positive definite.

Hence in Case 2 we are able to conclude that the sequence $\{w_k\}$ is uniformly bounded in the $C^{2,\beta}$-topology, hence in $C^k(\mathbb{R}^4)$ for all k.

Case 1 can be excluded by means of Theorem 10.3, which will be proven in Step 2. In fact, so far we know by (10.7) that $\bar{w}_k \to \bar{w}$ in $C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^4)$ with

$$\sigma_2(A_{\bar{g}_w}) = 0 \text{ and } \bar{w} \in C^{1,1}(\mathbb{R}^4), \tag{10.9}$$

$R_{\bar{g}_w} \ge 0$, where (10.9) is meant to hold in the weak sense, i.e., a.e. on $\mathbb{R}^4$, or in integrated form. Hence $\bar{w} \equiv$ const., in particular $\nabla \bar{w}(0) = 0$ contradicting (10.5).

Step 2. *Proof of Theorem* 10.3. Fix $B_\varrho :\, = B_\varrho(0)$, choose a cut-off function $\eta \equiv 1$ on $B_\varrho, \eta \equiv 0$ on $\mathbb{R}^4 \backslash B_{2\varrho}$ with $|\nabla \eta| \lesssim \varrho^{-1}, |\nabla^2 \eta| \lesssim \varrho^{-2}$, and set $\bar{w} :\, = \fint_{B_{2\varrho}} w \, dx$.

Multiply the expression (7.10) for $\sigma_2(A_{g_w})e^{4w}$, which holds a.e. on $\mathbb{R}^4$, by the function $(w - \bar{w})\eta^4$ and integrate on $\mathbb{R}^4$. Using the assumption of Theorem 10.3 one obtains

$$\int_{\mathbb{R}^4} |\nabla w|^4 \eta^4 \, dx \lesssim \left(\int_{A_\varrho} |\nabla w|^4 \eta^4 \, dx \right)^{\frac{1}{2}},$$

where $A_\varrho :\, = B_{2\varrho} - B_\varrho$. Since $\int_{\mathbb{R}^4} |\nabla w|^4 \, dx \le ||w||^4_{C^{1,1}} < \infty$, we have

$$\lim_{\varrho \to \infty} \int_{A_\varrho} |\nabla w|^4 \eta^4 \, dx = 0,$$

hence $\lim_{\varrho \to \infty} \int_{B_\varrho} |\nabla w|^4 \, dx = 0$, i.e. $|\nabla w| \equiv 0$ on compact subsets of $\mathbb{R}^4$, which implies that $w \equiv$ const. □

Notice that this proof works also in the case when $\sigma_2 \equiv \varepsilon << 1$, which will be used in the degree-theoretic argument later.

Step 3. *Proof of Theorem* 10.4. We recall the geometric proof of Obata's Uniqueness Theorem on S^n: If $R_g \equiv$ const. on S^n, then $|E| \equiv 0$ and $g = \phi^*(g_c)$ for some conformal transformation $\phi\colon S^n \to S^n$. For simplicity we review Obata's proof for $n = 4$. Then $E_{ij} = -2u^{-1}(\nabla_g^2 u)_{ij} + \frac{1}{2}u^{-1}(\Delta_g u)g_{ij}$, where $g = u^2 g_0$, and calculating in the g metric ($dv\colon = dv_g$),

$$\begin{aligned}\int_{S^4} |E|^2 u\,dv &= \int_{S^4} g(E,E)u\,dv\\ &\underset{(TrE=0)}{=} -2\int_{S^4} g(E,\nabla_g^2 u)\,dv\\ &= 2\int_{S^4} g(\delta E, du)\,dv\\ &\underset{(\delta E=\frac{1}{4}dR)}{=} 2\int_{S^4} g\left(\frac{1}{4}dR, du\right)\,dv \underset{(R\equiv\text{const.})}{=} 0.\end{aligned}$$

On $\mathbb{R}^4$, and assuming $R_g \equiv$ const., we use a cut-off function to imitate Obata's proof:

$$\begin{aligned}\int_{\mathbb{R}^4} g(E,E)u\eta^2\,dv &= -2\int_{\mathbb{R}^4} g(E,\nabla_g^2 u)\eta^2\,dv\\ &= \int_{\mathbb{R}^4} g(\delta E, du)\eta^2\,dv + 2\int_{\mathbb{R}^4} g(E,du)\nabla_g(\eta^2)\,dv\\ &\underset{(R_g\equiv\text{const.})}{\le} 2\int_{A_\varrho} |E|_g|\nabla_g u||\nabla_g(\eta^2)|\,dv\\ &\lesssim \left(\int_{A_\varrho} |E|_g^2 u\eta^2\,dv\right)^{\frac{1}{2}}\left(\int_{A_\varrho} |\nabla_g u|^2|\nabla_g\eta|^2 u^{-1}\,dv\right)^{\frac{1}{2}}.\end{aligned}$$

Hence it suffices to prove

$$\begin{aligned}\int_{A_\varrho} |\nabla_g u|^2|\nabla_g\eta|^2 u^{-1}\,dv &= \int_{A_\varrho} |\nabla_0 u|^2|\nabla_0\eta|^2 u^{-1}\,dx\\ &\le C \text{ independent of } \varrho.\end{aligned} \tag{10.10}$$

Since then (as before) $E \equiv 0$ follows by taking $\varrho \to \infty$. To prove (10.10) one may look at the situation for general n, and (10.10) amounts to showing that

$$I(\varrho)\colon = \frac{1}{\varrho^2}\int_{A_\varrho} |\nabla_0 u|^2 u^{-1}\,dx$$

is bounded independent of ϱ. For $n = 3$ this can easily be done by multiplying the differential equation $-\Delta_0 u = c_3 u^{\frac{n+2}{n-2}} (= c_3 u^5)$ by $u^{-\frac{n-2}{2}}$ to get $I_3(\varrho) \le C$. If there is a volume bound, then one can easily check that $u^{-1} \le c|x|^2$ for all n, and it remains to show that

$$\int_{A_\varrho} |\nabla_0 u|^2\,dx \le C \quad \text{independent of } \varrho.$$

In general, a volume bound is too strong an assumption. For $n = 4$ in our situation we proceed with a similar strategy replacing R_g by $\sigma_2(A_g)$ and E by some tensor L with similar properties.

Lemma 10.8 *Suppose (M^4, g) is locally conformally flat (e.g., for $g = e^{2w}|dx|^2$), then consider the tensor*

$$L\colon = \frac{1}{4}|E|^2 g + \frac{1}{6}RE - E^2.$$

Then

$$\begin{cases} Tr_g L &= 0, \\ \delta L &= \frac{1}{2} d\sigma_2(A). \end{cases} \tag{10.11}$$

Proof. Follows from a straightforward computation. □

Proposition 10.9 *If $\sigma_2(A) > 0, R > 0$, then*
(i) *$g(L, E) \geq 0$ with equality iff $E \equiv 0$,*
(ii) *$|L|^2 \leq \frac{R}{3} g(L, E)$.*

Proof. (i) is a consequence of the relation $TrE^3 \leq \frac{1}{\sqrt{3}}|E|^3$, which was already used in (8.17).
(ii) One calculates

$$|L|^2 = |E^2|^2 - \frac{1}{4}|E|^4 + \frac{1}{36}R^2|E|^2 - \frac{1}{3}RTrE^3,$$

and $|E^2|^2 \leq \frac{7}{4}|E|^4$, which is sharp, since E might have diagonal form $(E_{ij}) =$

$$\begin{pmatrix} -3\lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}.$$

□

Now we can proceed to sketch a proof of Theorem 10.4 along the lines of Obata's proof outlined above.

$$\begin{aligned}
\int_{\mathbb{R}^4} g(L, E) u \eta^4 \, dv_g &\underset{(10.11)}{=} -2 \int_{\mathbb{R}^4} g(L, \nabla_g^2 u) \eta^4 \, dv_g \\
&= 2 \int_{\mathbb{R}^4} g(\delta L, du) \eta^4 \, dv_g + 2 \int_{\mathbb{R}^4} g(L, du) \nabla_g(\eta^4) \, dv_g \\
&\underset{\substack{(\sigma_2 \equiv \text{const.}) \\ (10.11)}}{\leq} 8 \int_{\mathbb{R}^4} |L|_g |\nabla_g u| |\nabla_g \eta| (\eta)^2 \, dv_g
\end{aligned}$$

$$
\begin{aligned}
&\underset{(ii)}{\leq} \frac{8}{\sqrt{3}} \int_{\mathbb{R}^4} R^{\frac{1}{2}} g^{\frac{1}{2}}(L,E) |\nabla_g u| |\nabla_g \eta| (\eta)^2 \, dv_g \\
&\lesssim \left(\frac{1}{\varrho^2} \int_{A_\varrho} R |\nabla_0 u|^2 u^{-1} \, dx \right)^{\frac{1}{2}} \left(\int_{A_\varrho} g(L,E) u \eta^4 \, dv_g \right)^{\frac{1}{2}} .
\end{aligned}
$$

Thus it suffices to prove that there is a constant C independent of ρ, such that

$$
\int_{A_\varrho} R |\nabla_0 u|^2 u^{-1} \, dx \leq C \varrho^2, \tag{10.12}
$$

since then arguments analogous to Obata's proof show that $g(L,E) = 0$, which by Proposition 10.9 (i) implies $E \equiv 0$.

In order to show (10.12) one multiplies the expression (7.10) for $\sigma_2(A_g)e^{4w}$ by e^{-w}, which leads to (10.12) for $n = 4$. Also for $n = 5$ this can be worked out, but this method seems to fail for $n \geq 6$. □

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Sun-Yung Alice Chang

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Non-linear elliptic partial differential equations are an important tool in the study of Riemannian metrics in differential geometry, in particular for problems concerning the conformal change of metrics in Riemannian geometry. In recent years the role played by the second order semi-linear elliptic equations in the study of Gaussian curvature and scalar curvature has been extended to a family of fully non-linear elliptic equations associated with other symmetric functions of the Ricci tensor. A case of particular interest is the second symmetric function of the Ricci tensor in dimension four closely related to the Pfaffian.

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ISBN 3-03719-006-X

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